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THE MATHEMATICS TEACHER

VOLUME XVI MAY 1923 No. 5

EMPIRICAL THEOREMS IN DIOPHANTINE ANALYSIS.

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The larger portion of the theorems in Diophantine Analysis probably existed first as empirical or conjectural theorems. Many of them passed to the state of proved theorems before they left the hands of those who discovered them; many others were proved in the same generation in which they were made public; not a few required a longer period for their proof; and several remain today as a silent challenge to the skill and power of contemporary mathematicians. The remarks may be illustrated with a brief account of the history of the problem of representing numbers (that is, positive integers) as sums of squares of integers and of higher powers. Anyone interested in further details will find them in the comprehensive account of Diophantine Analysis which fills volume II (*xxvi* + 803 pages) of L. E. Dickson's "History of the Theory of Numbers," Carnegie Institution, Washington, D. C. We shall make free use of the material summarized in a masterly way in this volume.

In the problem of the representation of numbers as sums of two squares we have the fundamental identity:

$$(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2.$$

This was known to the Greek mathematician Diophantus. The latter knew also that no number of the form $4n - 1$ is a sum of two squares. In 1625 A. Girard had already determined the numbers which are expressible as sums of two integral squares: a number is a sum of two squares if, and only if, its quotient by the largest square dividing it is a product of primes of the form $4n + 1$ or the double of such a product. Fermat called the theorem that every prime of the form $4n + 1$ is a sum of two squares (called Girard's theorem by Dickson, *l. c.*) the fundamental theorem on numerical right triangles; he stated (1654, 1658, 1659) that he had a proof of it by his method of indefinite descent. A proof by such a method is given on pp.

39-41 of my "Diophantine Analysis" in a sequence of lemmas the proofs of which are independent of the preceding exposition. The first proof was given by Euler in 1749 after an effort of several years. Several other proofs were given by Euler and Lagrange in the next 34 years. The difficulty of the theorem may be seen from the fact that it lay without published proof for 124 years. For a detailed history of the whole problem see Dickson, l. c., pp. 225-257.

Diophantus knew that no number of the form $8m + 7$ is a sum of three squares. Fermat stated that a number is a sum of three squares if, and only if, it is not of the form $4^n(8m + 7)$. Euler labored for many years to prove this theorem; Lagrange found a partial proof; Legendre gave a complete proof in 1798. The problem has since been treated by many writers. See Dickson, l. c., pp. 259-274.

Every positive integer is a sum of squares of four integers. It would be difficult to find a more elegant theorem than this in the whole range of mathematics. Bachet and Fermat ascribed to Diophantus a knowledge of the theorem on the ground that he expressed 5, 13 and 30 as sums of four rational squares in two ways without naming any necessary condition for representation, as a sum of four squares whereas he did name such conditions in dealing similarly with sums of two and of three squares. It seems to me that the evidence is hardly convincing. But the theorem was clearly and explicitly conceived by Bachet who verified it in 1621 for integers up to 325. Girard asserted the theorem in 1625, and Descartes mentioned it in 1638 as unproved. Fermat stated in 1659 that he had had much trouble in applying his method of indefinite descent to the proof of this theorem, but that he had finally succeeded; he did not make known his proof.

For more than forty years Euler sought a proof of the theorem. After twenty years of study he published in 1751 his first important facts bearing on the problem, including the identity which expresses as the sum of four squares the product of two such sums. The first proof was given by Lagrange in 1772; in his method he utilized certain ideas of Euler's paper. In 1773 Euler published an elegant and much simpler proof which (according to Dickson) has not been improved upon to date. Outlines of the proofs of Lagrange and Euler are given

by Dickson, l. c., pp. 279-282. The whole history is recorded in pp. 275-303.

The proof of such a theorem does not settle all of the interesting questions. We cannot avoid the inquiry as to the number of representations of a given integer n as a sum of four squares. We are prepared to expect that the discovery of the answer to the question and its proof will be much more difficult than the proof that there is at least one such representation. In fact the answer to the question was first found by bringing into connection with the problem certain striking results in the theory of elliptic functions. Certain remarkable identities among functions makes it possible to equate coefficients obtained in two ways one of which shows that the nth coefficient is equal to the number of ways in which n may be represented as a sum of four squares while the other gives a fairly direct means for finding the actual value of the coefficient. The result emerging is due to Jacobi. It may be stated as follows: Let p be odd and let $\sigma(p)$ be the sum of the divisors of p; then the number of representations of $2^a p$ is $8\sigma(p)$ or $24\sigma(p)$ according as a is or is not zero, where in a representation the signs of the roots and their order are taken into account. It must be admitted that this method of counting the representations is artificial; it is the method imposed by the means of proof afforded by the identities from the theory of elliptic functions. Several elementary proofs of the theorem of Jacobi have been given, one as late as 1914. A proof by means of theta functions was given in 1915.

In the presence of the theorem that every positive integer is a sum of four squares it is natural to raise the question as to whether every positive integer can be represented as a sum of a given number of non-negative cubes, of a given number of non-negative fourth powers, ... , of a given number of non-negative nth powers. In 1770 E. Waring stated, as an empirical theorem, that every positive integer is a sum of at most 9 positive cubes, of at most 19 positive biquadrates, and so on. It seems to be implied that a function $h(n)$ of n exists such that every positive integer is a sum of at most $h(n)$ nth powers; but Waring does not give a general formula for $h(n)$. Euler (about 1772) stated that it requires at least as many as

$$[(3/2)^n + 2^n - 2]$$

nth powers for the representation of some numbers m as a sum

of positive nth powers, where $[x]$ denotes the integral part of x; in fact such a number is

$$\{[(3/2)^n]-1\}2^n+2^n-1.$$

For $n=2, 3, \ldots, 8$, Euler's formula (1) gives the numbers 4, 9, 19, 37, 73, 143, 279 suggesting that it may be just the formula which Waring had in mind, if indeed he had any definite formula in mind. For a history of this subject see Dickson, l. c., pp. 717-729 and G. H. Hardy's Oxford Inaugural Lecture, 1920.

For convenience in discussion let us define two symbols. We define $g(n)$ to be the least number, if such a number exists, for which it is true that every number is the sum of at most $g(n)$ nth powers. We define $G(n)$ to be the least number, if such a number exists, for which it is true that every number *from a certain point onwards* is a sum of at most $G(n)$ nth powers. For each of these numbers we desire two things: a proof of its existence; a determination of its value. From the results which we have already stated it follows readily that $g(2)=G(2)=4$. It is clear that the existence of either of the numbers $g(n)$ and $G(n)$ implies that of the other. Their existence for general n was first established by Hilbert in 1909; but his method of proof gave no satisfactory information concerning the value of either $g(n)$ or $G(n)$. It has, however, been established that $g(3)=9$ but it is yet unknown whether $g(4)$ is or is not equal to 19.

Let us consider further the problem for cubes. While it is known that $g(3)=9$ there are only two known numbers requiring as many as nine cubes each for their representation as a sum of cubes, namely 23 and 239; and it has been conjectured that every positive integer except 23 and 239 is a sum of at most 8 positive cubes. Landau has *proved* that the number of numbers requiring as many as 9 cubes is *finite*. From this it follows that $G(3)\leqq 8$. Since every cube is congruent to 0, or 1, or -1, modulo 9, it follows that the sum of 3 cubes cannot be of the form $9m+4$ or $9m+5$. Hence $G(3)\geqq 4$. Therefore the value of $G(3)$ is one of the numbers 4, 5, 6, 7, 8; but it is still undetermined which of these is the true value of $G(3)$. Thus we know the value of $g(3)$, but there is a wide range of uncertainty about the value of $G(3)$. There is much empirical evidence for the conjecture that $G(3)\leqq 6$.

For the representative of each of the numbers 79, 159, 239, 319, 399, 479*, 559 as a sum of biquadratics it is necessary to use 19 summands. Hence $g(4) \gtreqq 19$. It has been proved that $g(4) \leqq 37$. Thus $g(4)$ is one of the numbers 19, 20, . . . , 36, 37; but it is unknown which of them gives the true value of $g(4)$. It is known that $G(4)$ is confined to a much narrower range and that its true value is one of the numbers, 16, 17, 18, 19, 20, 21.

In 1919 G. H. Hardy and J. E. Littlewood proved that $G(k) \leqq k.2^{k-1} + 1$. They have later improved this result by showing that $G(k) \leqq (k-2)\, 2^{k-1} + 5$. In their work they have made extensive and remarkable use of the theory of analytic functions, introducing methods which seem certain of having a great influence on the development of the subject. It is known also that $G(k) \gtreqq k + 1$, while if k is a power of 2 (higher than the first) we have $G(k) \gtreqq 4k$.

Results of this sort seem far enough separated from the famous conjecture of Goldbach in 1742 that every even number is a sum of two odd primes or from the similar conjecture that every odd number (at least from a certain point onwards) is the sum of three odd primes; but the methods of Hardy and Littlewood for dealing with the former problems have enabled them to open up the attack on the latter. They have not been able indeed to prove the Goldbach theorem nor to establish the other except on the assumption of "the truth of the notorious Riemann hypothesis concerning the zeros of the Zeta-function, and indeed in a generalized and extended form." The Goldbach theorem for even numbers stands before us still as a challenge to our ingenuity. It remains unproved after about 180 years; it has not even been proved that every even number m is a sum of any given number of primes, the number being independent of m.

Euler was not able to prove the Goldbach theorem for even numbers, though he believed it to be true. Descartes stated that every even number is a sum of 1, 2 or 3 primes. Waring (in 1770) reproduced the statements of Goldbach that every even number is a sum of two primes and that every composite

*In the sixth line from the bottom of p. 22 of an article by C. A. Bretschneider in Crelle's Journal, vol. 46 (1853), this number is incorrectly printed 379; and the error is reproduced on page 717 of Dickson's History, vol. II.

odd number is a sum of three primes. Euler stated (in 1775) without proof the theorem that every number of the form $4n + 2$ is a sum of two primes each of the form $4k + 1$, and verified this for values $4n + 2$ up to 110. A number of related theorems of this sort (some of which we shall give below) have been announced; but they stand before us unsolved as a perpetual reminder of our inability to deal with certain problems having a very simple formulation.

On pages 421-425 of volume I of Dickson's *History* we have a record of the conjecture of the following theorems: Every odd number is the sum of a prime and the double of a square (verified by Euler up to 2500); every number of the form $8n + 3$ is the sum of an odd square and the double of a prime $4n + 1$ (verified by Euler up to 187); every prime $4n—1$ is the sum of a prime $4m+1$ and the double of such a prime (a conjecture which I have verified for primes up to 1300); every even number is the difference of two primes (and perhaps of two consecutive primes) in an infinitude of ways; every odd number greater than 3 is of each of the forms $p_1 + 2p_2$, $p_1 — 2p_2$, $2p_1 — p_2$, where p_1 and p_2 are primes; every multiple of 6 is a difference of two primes of the form $6n + 1$. I have also verified up to 1300 that primes of the form $16n + 5$ may be represented in the form $4p + q$ and the primes of the form $36n + 7$ may be represented in the form $6p + q$, where p and q are primes of the form $4k + 1$.

In the second volume of Dickson's *History* we have a record of the following related conjectures: The triple of an odd square not divisible by 5 is a sum of squares of three primes other than 2 and 3 (p. 266); the double of any odd integer is a sum of two primes $4n + 1$ (pp. 282, 289); every prime $18n \pm 1$ or else its triple is expressible in the form $x^3 — 3xy^2 \pm y^3$ (p. 575).

Some of the conjectural theorems mentioned in the two preceding paragraphs are of very great importance, if true. Let us take the theorem which asserts that every prime of the form $4n — 1$ is a sum of a prime of the form $4m + 1$ and the double of such a prime. If we assume the truth of this theorem we can readily prove (by aid of well known results having a simple and elegant demonstration) that every integer is a sum of at most four squares. We shall now give such a proof:

From the formula $2(a^2+b^2)=(a+b)^2+(a-b)^2$ it follows that the double of a sum of two squares is itself a sum of two squares. Now every prime of the form $4m+1$ is a sum of two squares, as we have already seen; and hence the double of such a prime is a sum of two squares. Hence if a prime $4n-1$ is a sum of a prime $4m+1$ and the double of such a prime it is also a sum of four squares. But a prime $4n+1$ is a sum of two squares; and the prime 2 is a sum of two squares. Hence every *prime* number is a sum of at most four squares since every odd prime number is of one of the forms $4n+1$, $4n-1$. Now from the Euler formula

$$(a^2+b^2+c^2+d^2)(p^2+q^2+r^2+s^2)=x^2+y^2+z^2+v^2,$$

where

$$\begin{aligned} x &= ap+bq+cr+ds,\\ y &= aq-bp\pm cs\mp dr,\\ z &= ar\mp bs-cp\pm dq,\\ v &= as\pm br\mp cq-dp, \end{aligned}$$

it follows readily that the product of numbers each of which is a sum of at most four squares is itself a sum of at most four squares. But we have already seen that any given prime number is a sum of at most four squares. Therefore every number has this property. Thus we have a ready proof of the theorem that every number is a sum of at most four squares if we grant the conjectural theorem in consideration. But no way is known to derive the latter from the former. It appears therefore that the latter is probably the more deep-lying theorem.

In view of the great importance of this theorem it seems desirable to have a wide range of empirical evidence for it, since we are still unable to give a logical proof. If some reader of this journal, having access to some such table of primes as that of Lehmer, should verify the theorem for a wide range of values the result would be of interest to the present writer and perhaps to a considerable number of people working in this field.

A similar importance attaches to the conjectural theorem that the double of any odd integer is a sum of two primes of the form $4n+1$. For if this is true and if $2k+1$ is any odd number we have

$$2(2k+1)=\alpha^2+\beta^2+\gamma^2+\delta^2.$$

Then the number of even integers in the set α, β, γ, δ is even. Hence we may choose the notation so that α and β are both odd or

both even, and γ and δ are both odd or both even. Then we have

$$4(2k+1) = (\alpha+\beta)^2 + (\alpha-\beta)^2 + (\gamma+\delta)^2 + (\gamma-\delta)^2,$$

where the second number is a sum of four even squares. Dividing each term of the equation by 4, we obtain a representation of $2k+1$ as a sum of at most four squares. Since 2^a also has this property we employ the foregoing identity of Euler to prove that every positive integer is a sum of at most four squares.

A great deal of attention has been given in recent years to the Last Theorem of Fermat, stated by Fermat in the following form: "It is impossible to separate a cube into two cubes, or a biquadrate into two biquadrates, or in general any power higher than the second into two powers of like degree." On the margin of a book which he was reading Fermat added: "I have discovered a truly remarkable proof which this margin is too small to contain." Thus Fermat asserts that he has a proof of the impossibility of the equation

$$x^n + y^n = z^n$$

in positive integers when n is a given integer greater than 2. This theorem remains unproved to the present day.

One may raise the question for the relevant investigation in a form somewhat different from that in which it is usually put. Thus we may ask what is the value of the function $f(n)$ which is such that $f(n)$ denotes the smallest number such that the sum of $f(n)$ positive nth powers may itself be an nth power. Not much is known about this function $f(n)$. From the theory of Pythagorean triangles it is seen that $f(2)=2$. Since 2^n is equal to a sum of 2^n units it is clear that $f(n) \leqq 2^n$; but this is certainly much too large a limit.

It is known that $f(3)$ has the value 3. For it has been proved that the sum of two cubes cannot be a cube, while $9^3 = 8^3 + 6^3 + 1^3$, showing that the sum of three cubes may be a cube.

It is known that the sum of two fourth powers cannot be a fourth power; hence $f(4) > 2$. On the other hand the example (Dickson, vol. II, p. 652)

$$353^4 = 30^4 + 120^4 + 272^4 + 315^4$$

shows that $f(4) \leqq 4$. It has been conjectured that $f(4) > 3$, that is, that a sum of 3 fourth powers cannot be a fourth power; if this is true it follows that $f(4) = 4$, but this fact (I believe) has never been established.

In general, it has been conjectured that $f(n)$ is never less than n. But this has been *proved* only for $n = 2, 3$. For $n > 4$ no numerical relation seems to be known implying that $f(n)$ is less than $n + 1$. The relation

$$4^5 + 5^5 + 6^5 + 7^5 + 9^5 + 11^5 = 12^5$$

has been found; and this shows that $f(5) \leqq 6$. It is also known that $f(5) \gtreqless 3$. But it appears not to have been determined which of the numbers 3, 4, 5, 6 affords the true value of $f(5)$. When we pass to exponents greater than 5 the known results are still more fragmentary.

To the foregoing list of conjectured theorems in Diophantine Analysis we shall add one other of a different sort. Consider the set of numbers $m_1, m_2, m_3, \ldots$ where $m_1 = 1$ and $m_{k+1} = m_k(m_k + 1)$. We have $m_1 = 1$, $m_2 = 2$, $m_3 = 6$, $m_4 = 42$, $m_5 = 1806, \ldots$ It is easy to show that

$$\frac{1}{m_1 + 1} + \frac{1}{m_2 + 1} + \cdots + \frac{1}{m_{n-1} + 1} + \frac{1}{m_n} = 1, \quad n \gtreqless 2,$$

the method of proof being by induction. In terms of the numbers m_k it is then easy to write down a solution of the Diophantine equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} = 1.$$

It has been conjectured by O. D. Kellogg that the maximum value of an x which can occur in a solution of this equation (for fixed n) is m_n. (See my Diophantine Analysis, p. 115; American Mathematical Monthly, 28 (1921), p. 77; and a forthcoming paper by Professor Kellogg in the American Mathematical Monthly. Since this article was sent to the editors Kellogg's Diophantine equation has been treated successfully by D. R. Curtiss in *American Mathematical Monthly,* 29 (1922), pp. 380-387.) It seems to be difficult to find a proof of this conjectured theorem of Kellogg. If the theorem is true a proof of it will lead readily to a complete and interesting theory of the given Diophantine equation.

THE PENNSYLVANIA STATE COURSE OF STUDY IN MATHEMATICS

By JOHN A. FOBERG
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The State Course of Study in Mathematics in Pennsylvania is a twelve-year program. This program is continuous not only in the sense that it is laid down year by year, beginning at the first school year and extending through the twelfth school year, but also in the sense that it is clearly understood that each subject after it has once been introduced is thereupon to be regarded as an element entering into all later instruction in mathematics throughout the course of study. Thus, for instance, arithmetic is looked upon as an implicit, if not an explicit, element in the mathematics program of studies throughout the junior and senior high school. So, also, after any element of algebra or geometry is taken up in class work, it is understood that this element is to be regarded as part and parcel of the subject matter of instruction for all later semesters and years.

The School Code of Pennsylvania recognizes the 6-3-3 type of organization wherever the Junior-Senior High School is organized, and the 8-4 type of organization where the Junior High School does not appear as part of the high school organization.

Mathematics of the elementary school,—that is, of years 1-6 inclusive,—is devoted to the study of arithmetic. The objectives that are set up are a reasonable degree of speed and accuracy and facility in the fundamental operations applied to integers and fractions, common and decimal. The attainment of these objectives, regularly, in the instruction of the elementary school, is brought nearer by the large amount of study and investigation that is now being devoted to the problems of learning and teaching arithmetic. As we succeed, in larger and larger measure, in getting the results of these studies to the students in our normal school classes, we hasten the day when desirable conditions now attained in a few experimental schools will become widespread. These skills and facilities will then be the

secure foundation upon which we shall build the work of the junior high school.

Inevitably a school program that looks to progressive improvement in teaching any subject, whether in the elementary school or in the high school, must concern itself with the preparation of the teachers who are to do the classroom work. The courses of study in Pennsylvania's normal schools are now being carefully considered, and it is hoped and expected that revision will result in better provision for instruction in both the content and method of teaching of mathematics in the elementary school. At the present time, the only requirement for qualification as teacher of mathematics in our high schools is the satisfactory completion of twelve semester hours of work in standard college courses in mathematics. It is felt by many that a more satisfactory minimum requirement would be the completion of at least a semester's work in the calculus. Revision in the direction of more rigorous requirements for teachers of high school mathematics must of necessity be attained slowly.

There is at the present moment a very large amount of keen interest in the study of the problems tied up with instruction in mathematics in the high schools of the state. Perhaps the largest amount of definitely organized interest and endeavor is associated with the problems of the junior high school. In Pennsylvania, as elsewhere in the United States, the junior high school is developing with a considerable degree of momentum. A large number of communities either have already taken steps to inaugurate junior high schools or are seriously contemplating the wisdom of so doing. Wherever the junior high school is started it is relatively easy to secure recognition of the need for reorganizing the procedure of mathematics instruction for years 7, 8, and 9. Such recognition is far easier to secure with the new type of school than it is for the corresponding years and classes of pupils in the conventional 8-4 types of school. In a number of places in the state work of a high degree of merit is already being done in the junior high school. There is promise that as time goes on the junior high school will react upon the work of the elementary school and upon the work of the senior high school as well, leading to increased effectiveness in the mathematics instruction in both of these school-levels.

The mathematics of the junior high school,—that is of years 7-9 inclusive,—is conditioned by the nature of the junior high school itself. This school organization is looked upon as affording a try-out period in which the pupil tests himself and the subject matter of instruction to the end that he may discover his own aptitudes, interests, and abilities. In view of this try-out character of the junior high school, it follows of necessity that the subject matter of instruction in each one of the branches of instruction must be of as widely distributed and general a nature as it is possible to compass. To be specific, in mathematics it will not do to confine the subject matter of instruction to arithmetic alone in years 7 and 8, as has been customary heretofore, or to algebra alone in the ninth year, as is the case in conventional programs of study. This confining of the subject matter of instruction to a very limited field in each one of the branches defeats the purpose for which the junior high school is set up; that is, affording the pupil as wide an opportunity as can possibly be managed for contact with each one of the various fields of learning.

The mathematics of years 7, 8, and 9 is planned to include the fundamental notions of arithmetic, algebra, geometry, and numerical trigonometry. It is assumed that the mathematics of these three years is to be required of all pupils in the junior high school, and furthermore that the mathematics of the ninth year will be the last mathematics required of all pupils. It follows that the material entering into the junior high school mathematics course of study must be such as is defensible on the ground of its value to all students, as preparation for citizenship. Complications rarely met with in practical applications of mathematics to real situations, and the development of skills that are unlikely to have opportunity for exercise in later work, are not to be included in the work of the junior high school.

The most desirable arrangement in sequence of the materials of instruction is a matter that has not yet been settled on the basis of classroom experience. Much experimental work must be done before this question is finally settled. In the meantime, the schools are encouraged to try various arrangements, and various sets of textbooks, to the end that experience may accumulate and enable us ultimately to arrive at a conclusion as to the best procedure.

Interesting problems are involved in the junior high school course of study in mathematics by reason of the fact that frequently part of the seventh and eighth year classes are included in conventional eight-year elementary schools and part are organized in the junior high school. Wherever this is the case it is obvious that the senior high school receives two streams of entering pupils; one coming to its ninth year by way of the eight-year elementary school, and another stream coming to its tenth year by way of the junior high school. Naturally this implies that the tenth year pupils in the senior high school have not all had the same type of training, and unless effective adjustments are made in order to coordinate the training given by the senior high school in its ninth year and that given in the ninth year of the junior high school there is sure to be difficulty with the mathematics teaching of the tenth year. Happily, experience shows that it is possible to bring about the necessary adjustments to reduce the difficulties in this tenth year to a minimum.

Furthermore, it becomes highly desirable to reorganize the mathematics instruction of the seventh and eight years in the conventional eight year elementary school, to bring it in line with the program of the junior high school. It seems reasonable to grant that if certain types of materials and methods of instruction are appropriate to children in their seventh and eighth years of school life in a junior high school, then the same types of subject matter and method of instruction are likewise appropriate to children of the same age who happen to be in conventional eight-year elementary schools.

The mathematics of the commercial course in which is enrolled so large a fraction of our high school population presents another question that is highly important from the point of view of the program as a whole. The conventional arrangement involves the replacing of the general mathematics course of the junior high school by technical work in commercial arithmetic at some place in the junior high school course of study. Frequently this substitution is made at the middle of the eighth year or at the beginning of the ninth year. Experience seems to show that commercial arithmetic studied as an isolated topic in the ninth year does not afford a sound basis for the mathematics that is needed in connection with the bookkeeping courses

which usually come in the tenth year of the commercial courses. It would seem that it is much more desirable to have all junior high school pupils, commercial pupils as well as others, pursue the same mathematics course of study through the ninth year. Following this ninth year of general mathematics study, commercial pupils will more successfully take up the study of the technical applications of mathematics to business procedure in connection with the bookkeeping, than by taking their commercial arithmetic in the ninth year in anticipation of a demand for the skills therein developed in connection with the tenth year bookkeeping work.

The mathematics of the senior high school is elective throughout and will therefore be taken by those pupils only who have well defined purposes in view in electing it. These purposes will be based upon a felt need for mathematics as a requirement for work to be done after completing the high school course of study or upon a real liking for mathematics and appreciation of its value in the life of today. Under typical circumstances the mathematics of the tenth year will be devoted largely to plane geometry, the eleventh year will be divided between algebra and solid geometry, and the twelfth year given up partly to plane trigonometry and partly to elective courses designed to serve the needs of the particular group of students as they may be discovered. It is thought that in some communities the twelfth year may very well contain work in the mathematics of finance for such pupils as contemplate entering at once upon business careers after completing the high school course of study. In other communities it may be that some type of engineering mathematics will best serve the needs of the pupils who are completing the high school course of study. Shop mathematics will sometimes be the best type of work with which to follow plane geometry, or even to replace it in the tenth year.

A large amount of freedom for the adaptation of the course of study for each of the individual high schools is left in the state course of study to the end that there may be the largest measure possible of experiment and initiative on the part of the various high schools. It is intended that as various arrangements of the material of instruction, or as various types of instruction are tried out, the results of such experiences may

be pooled and made available to all the other high schools in the state and thus be made to contribute as largely as possible to the advantage of mathematics instruction in the state as a whole.

It is fully appreciated that a large number of high schools will continue to follow the traditional course of study, devoting the ninth year to algebra alone, followed by the study of plane geometry. It is possible largely to increase the value of this arrangement of the mathematics program by simplifying it, through the elimination of elements not now considered important for pupils beginning the study of mathematics and by emphasizing the topics of outstanding importance. With this in mind, the State course of study recommends omitting or deferring such work as long division and multiplication involving polynomials of more than three terms; factoring of types other than the very simple ones, the omission of simultaneous equations involving more than two unknowns, and so on. By centering the study of algebra about such topics as the equation, the formula and graphic representation, it is hoped to secure the element of emphasis and relief that will save the subject from the monotony and lack of interest that accompanies the study of a subject in which all topics are upon precisely the same level of interest and importance.

An especially noteworthy feature in connection with interest in the reorganization of the mathematics course of study is the cooperative activity of groups of teachers and principals that has been called into play in a number of places. In two districts in the State, large groups of teachers and principals have met for study of the problems of instruction and administration in the junior high school, under the leadership of various members of the staff of the Department of Public Instruction. In addition, a number of college and university extension classes are being conducted, devoted wholly or in part to the study of the reorganization of mathematics instruction. The stimilus given to the professional education of teachers in service, through the provisions of the Edmonds Act embodying the educational policy of the State, has resulted in a very large amount of effective study by teachers in the summer courses of Pennsylvania's colleges and normal schools, and in summer schools generally. This work by teachers in service has already resulted

in a noticeable rise of the level of classroom instruction in many districts, and contains much promise for the State as a whole.

In the senior high school the requirements of the College Entrance Examination Board's examination play a large part in determining procedure in many places. The fact that the College Entrance Examination Board has recently inaugurated a movement which will tend somewhat to simplify its mathematics examinations and to reorganize materials to be involved in those examinations, is a hopeful augury for a reorganization of mathematics instruction that will lead to a better type of work in the senior high school than has been prevalent heretofore.

It is widely felt that the authorities in charge of the mathematics teaching in the colleges would do a real service in the cause of improved teaching in high schools by letting it be known that satisfactory work done in the ninth year of the high school, along the line of the general courses set forth in the report of the National Committee on Mathematical Requirements, will be accepted as a substitute for the traditional ninth year work in algebra. Many high school authorities are deterred from venturing upon any change in accustomed procedure by fear of loss of credit toward college entrance requirements, due to lack of explicit college approval of the proposed new procedure. It would seem that such approval on the part of college teachers of mathematics would certainly be given as soon as it is understood that the innovations proposed are along the safe and sane lines advocated by the National Committee. Negative replies to inquiries by high school authorities as to whether or not a college will give entrance credit for "general mathematics" or "composite mathematics" in the ninth year are probably largely due to failure on the part of the college folk to understand the meaning of the terms "general mathematics" and "composite mathematics." There is need for cooperation at this point, in the interests of more efficient mathematics work in the high schools generally.

The very large amount of interest that has been exhibited by the teachers of Pennsylvania in the state course of study and the intelligent appreciation that has been shown of the aims and objectives that are looked for as outcomes of instruction in

the various branches give reason to believe that in a comparatively short time it will be possible to point definitely to increased effectiveness in the high school teaching of our subject. Constructive suggestions from all sources, whether within or without the State, are eagerly welcomed and if the discussions at this present meeting result in such suggestions, the presentation of this paper will have been amply justified.

FATE AND FREEDOM[1]

By ALFRED KORZYBSKI

In this lecture I propose to analyze the principles on which the foundation of the Science and Art of Human Engineering must rest, if we are ever to have such a Science and Art.

As my aim is merely to offer a somewhat rude outline, I shall, as much as possible, avoid the use of such technical terms as would be essential to the precision demanded by a detailed presentation.

By Human Engineering I mean the Science and Art of directing the energies and capacities of Human Beings to the advancement of Human Weal.[2]

All human achievements are cumulative; no one of us can claim any achievement exclusively as his own; we all must use consciously or unconsciously the achievements of others, some of them living but most of them dead.

Much of what I will say has been said before by many others. It will be impossible to give a full list of authors but the names of a few stand prominent; two Englishmen, Alfred Whitehead and Bertrand Russell; one Frenchman, Henri Poincaré; one American, Professor Cassius J. Keyser; one German, Albert Einstein. I will largely use here their ideas, methods and language, as my main concern is the practical application of some of their great ideas. It would be very difficult to acknowledge fully all I owe to these authors; yet anyone acquainted with the literature of the subject will recognize my obligations, which are heavy.

The term Engineering, in its generally accepted meaning, I take as derived from the Latin ingenium, cleverness, that is, designing, constructing, building works of public utility. As a matter of fact, there does not as yet exist a science of human engineering. The semi-sciences such as sociology, economics, politics and government, ethics, etc., are supposed to deal with the affairs of man, but they are too hopelessly divided and have not as yet emerged from the mythological prescientific era.

If there is to be a science of human engineering, it must be mathematical in spirit and in method and if we do not possess

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[1] An address delivered before the joint meeting of the Detroit Mathematics and Detroit History Clubs, January 11, 1923; before the Mathematical Club of the University of Illinois, January 12; and at the University of Michigan, January 15, 1923.

[2] Manhood of Humanity, The Science and Art of Human Engineering, by Alfred Korzybski. E. P. Dutton, New York City.

methods to apply mathematical thinking to human affairs, such methods must be discovered. Can this be done?

Let us say a word about what has already been accomplished in this direction. The latest researches in the foundations of mathematics, chiefly accomplished by Whitehead, Russell, Poincare and Keyser, have disclosed the insufficiency and fallacies of the traditional logic and have produced an internal revolution in logic and mathematics. Mathematics and logic have been proved to be one; a fact from which it seems to follow that mathematics may successfully deal with non-quantitative problems in a much broader sense than was suspected to be possible.

Let me recall a delightful mathematical joke. A distinguished mathematician, I do not recall his name, produced some very pretty but very abstract mathematical work. He being intensely disgusted by the commercialization of science, wrote to a friend: "Thank God, I have finally produced something which will have no practical application." The irony of life is that a few years later, his discovery was applied to some branch of physics with great results.

That is what is happening now in another field. Engineers are getting hold of some of the latest, very general and very abstract, discoveries of mathematics and are trying, with increasing success, to apply them to the ordering and direction of human affairs.

Somewhere I read in a review of a book written by one of the scientists I have just named that, not all in the book is "real mathematics." I am not convinced that the writer of this review meant what he said. Every growth of mathematics, be it in the superstructure or in the deepening of the foundations, is "real mathematics," if those words are to have any significant meaning.

It is true that such familiar concepts as "sine," "cosine," "derivative," "integral," "graph," and the like, have, for the time being, a subordinate importance in human engineering; but, as I conceive it, mathematics is not limited to such concepts; it embraces many others, such as existence, class, type, dimension, order, limit, infinity (Cantor), non-existence of metaphysical infinitesimal (Weierstrass), invariant, variable, propositional function (Russell), doctrinal function (Keyser), the physico-

mathematical theory of events and of objects (Whitehead) and the relativity of space and time (Lorentz, Minkowski, Einstein, Whitehead,) etc. These concepts are of immeasurable import, for without them the foundations of human engineering could not be laid.

When I speak about the relativity of space and time, I do not refer to Einstein Theory alone. I use the term here in its broadest meaning as generally accepted in science, namely that absolute space and absolute time do not exist. The work of Einstein is very important, yet it seems to me that the theory of the relativity of space, time and *matter* as elaborated by Whitehead is more comprehensive and is more directly applicable for our immediate purpose.

Before proceeding further we will have to establish a vocabulary for our mutual understanding. Human engineering, if such a branch of science is to exist, must be democratic—dealing with all mankind, and its outline must be clear. I will sacrifice minute precision to general clarity.

No matter where we start, we must start with some undefined words which represent some assumptions or postulates. We see that knowledge at every stage presupposes knowledge of those undefined words. Let us call this fundamental fact the "circularity of knowledge." Words written or spoken and mathematical symbols are like signs, labels, which we attach to ideas, concepts corresponding to our experience.

"The concrete facts of nature are events exhibiting a certain structure in their mutual relations and certain characters of their own. The aim of science is to express the relations between their characters in terms of the mutual structural relations between the events thus characterized. The mutual structural relations between events are both spatial and temporal. If you think of them as merely spatial you are omitting the temporal element, and if you think of them as merely temporal you are omitting the spatial element. Thus when you think of space alone, or of time alone, you are dealing in abstractions, namely, you are leaving out an essential element in the life of nature as known to you in the experience of your senses. . . What I mean is that there are no spatial facts or temporal facts apart from physical nature, namely, that space and time are merely ways

of expressing certain truths about the relations between events. . . . To be an abstraction does not mean that an entity is nothing. It merely means that its existence is only one factor of a more concrete element of nature.[1]

The dynamic theory of "matter" alone (I omit other considerations) makes it obvious that we can not *recognize* an event because when it is gone, it is gone. Yet our daily experience tells us that amidst events there is something which is fairly durable, which we can recognize from day to day. Things which we can recognize are called objects. A label attached to an object is called a word. The meaning of a word is a complex notion; for our purpose we may say that the meaning of a word is actually or potentially given by a definition.

Here we must take into consideration a grave fact. The above mentioned mathematicians have introduced a new concept which they stress very justly. Not only do they distinguish between true and false propositions but also recognize the existence of statements which have the form of propositions, but which are neither true nor false, but are meaningless. These meaningless verbal forms should be of great practical concern because our daily language and even some would-be theoretical disciplines are interwoven with meaningless statements. It often happens that such a meaningless statement is designated by a special "noise" which can be reduced to a combination of letters giving it the semblance of a word. Obviously this noise is equally meaningless, even though volumes be written about it.

And now we are approaching the central problem of all human knowledge. A sign or a label, if attached to nothing is a pseudo-symbol which symbolizes nothing; that is, it is not a symbol at all but is merely a noise if spoken, or blotch of black on white if written. Before a sign may acquire meaning and therefore become a symbol there must *exist* something for this sign to symbolize. The problem of existence has several aspects and is extremely important though not all of these aspects concern us at this state. Poincaré defines *logical* existence as one free from contradiction. Russell derives existence from his theory of propositional function. "If φ (x) is sometimes true, we may say there are x's for which it is true, or we may say 'arguments

[1] The Concept of Nature. A. Whitehead. pp. 167, 168, 171.

satisfying φ (x) exist',". Russell's conception is much more fundamental, but for the time being, Poincaré's definition will be sufficient.

As we observed before, events, in the Whitehead sense, cannot be recognized, but the things we can recognize are called objects. An event is a very complex fact, and the relations between two events form an almost impenetrable maze. Events are recognized and labeled by the objects situated in them. Obviously an object is not the whole of the event, nor does the label which symbolizes the object cover the whole of the object. It is evident that everytime we mistake the object for the event we are making a serious error, and if we further mistake the label for the object, and therefore for the event, our errors become more serious, so serious indeed that they too often lead us to disaster. As a matter of fact, we all of us have from time immemorial indulged in this kind of mental stultification, and here we find the source of most of the metaphysical difficulties that still befog the life of man.

In his last book, "Mathematical Philosophy[1]," Professor Keyser stresses the importance of recognizing that mankind is under the rule of logical fate. The concept of Logical Fate seems to be self evident when stated; it essentially means that from premises consequences follow. But the moment this is analysed with a full awareness of the circularity of all human knowledge those few words gain the significance of a discovery and formulation of a neglected law of immeasurable importance. By laws I mean propositions asserting relations which have been or can be established by experiment or observation.

The few first words with which mankind started its vocabulary were labels for prescientific ideas, naive generalizations full of silent assumptions, objectifications of non-existents, and our ignorant ancestors began to impose upon nature their naive fancies, which were mostly arbitrary. Sad to say, we continue to do the same in a great many fields.

Our daily speech and in very large measure our scientific language is one enormous system of such assumptions. The moment assumptions are introduced, and it is impossible to avoid them, logical destiny begins its work; and if we do not go back all the time, uncover and discover our conscious or unconscious

[1]E. P. Dutton

fundamental assumptions and revise them, mental impasses permanently obstruct the way. The history of human thought gives us many examples. One single concept, one generalization, be it meaningless (dealing with non-existents) or loaded with significance, gives rise to whole systems of thought—absurd or wise. Most of the false theories in the world are not so deficient in their reasoning as in the assumptions and concepts about which they reason—concepts that are vague, false to facts and often deal with non-existents.

Allow me to give an example in the wording of Whitehead. This example alone is enough to emphasize the exceeding importance of mathematics in the clarification of our mental processes.

"Aristotle asked the fundamental question, What do we mean by 'substance'? Here the reaction between his philosophy and his logic worked very unfortunately. In his logic, the fundamental type of affirmative proposition is the attribution of a predicate to a subject. Accordingly, amid the many current uses of the term 'substance' which he analyzes, he emphasizes its meaning as 'the ultimate substratum which is no longer predicated of anything else'.

"The unquestioned acceptance of the Artistotelian logic has led to an ingrained tendency to postulate a substratum for whatever is disclosed in sense-awareness, namely, to look below what we are aware of for the substance in the sense of the concrete thing. This is the origin of the modern scientific concept of matter and of ether, namely they are the outcome of this insistent habit of postulation . . . what is a mere procedure of mind in the translation of sense-awareness into discursive knowledge has been transmuted into a fundamental character of nature. In this way matter has emerged as being the metaphysical substratum of its properties. . . . Thus the origin of the doctrine of matter is the outcome of uncritical acceptance of space and time as external conditions for natural existence . . . What I do mean is 'the unconscious presupposition of space and time as being that within which nature is set'."[1] Otherwise absolute space and absolute time.

It becomes clear now, that "logical destiny" is a law which works within us consciously or unconsciously. Our language as a whole may be regarded as a vast system of assumptions and

potential doctrines with *fixed logical boundaries.* It was built with the metaphysical background of metaphysical infinitesimals, metaphysical infinity, absolute space and absolute time. A great many of the most important terms like change, continuity, cause and effect, moment, duration, etc., present a not only perplexing but insoluble problem because of the silent assumption of the existence of those non-existents. With the mathematical clarification of a very few of such fundamental concepts we may confidently expect that many of our difficulties will vanish, that the universe will become correspondingly intelligible, and man correspondingly intelligent.

Professor Keyser's "doctrinal function" reveals the inherent structure of doctrines and, therefore, in a large measure, of language and teaches us the methods by which to judge and to revise them. The circularity of knowledge shows us the absolute necessity of constant revision of our assumptions.

Most of what I have said is hardly so much as a sketchy outline of a vast coherent system, due, in the main, to the recent work of the few mathematicians before mentioned. The sharp formulation by these thinkers of the conditions of knowledge and progress promise that the coming epoch will be more fruitful for man than any other recorded by history. When the mathematicians themselves digest this new material, they cannot fail to see their rôle clearly as the leaders of pure thought and consequently of human progress.

Thought, taken in its broad meaning, is a process. Man thinks with his *whole* being; this process is not clearly delineated; it starts somehow with hazy "instincts," "feelings," "emotions," and crystalizes itself in a concept. We cannot but see that any divisions that we make in the process called thinking, are *arbitrary* and often misleading, or even meaningless.

There are, however, two aspects of this great process with which we can deal in a rigorous fashion. I refer on the one hand to that great invariant called the laws of thought, and, on the other hand, to those crystalized products of thinking which we are wont to call concepts. We should not fail to note that, at the various stages of this process, there is a striking difference in respect to what may be called its velocity. The velocities of so-called instincts, intuitions, emotions, etc., are swift, like

[1] A. Whitehead: The Concept of Nature, pp. 16, 18 ff.

a flash, while the analysis of the raw material thus presented and the building out of it of concepts and speech is slow. In this difference of velocity lies, I suspect, the secret of "emotions," etc. Unexpressed, amorphous thought is somehow very closely connected with, if not identical with, emotions. We all know, if we will but stop to reflect upon it, how very slow is the crystalization and development of ideas.

It is useless to argue which comes "first," "human nature" or "logic." Such argument has no meaning. "Human nature" and "logic" have their common starting point in the physico-chemical changes occurring in man, and as such, start simultaneously. We are thus enabled to see the supreme importance of concepts, which, as before suggested, are crystals of thought. Such crystals once produced, are permanent and they serve to precipitate their kind from out the supersaturated solutions of the emotions.

It is now evident that intellectual life is one long process of abstractions, generalizations, and assumptions; the three things are so many aspects of one *whole* activity. These processes materialize in symbols which we call words. We see also that all intellectual life is one vast (probably infinite) system of doctrines and doctrinal functions in the making, inherently governed by logical fate. As Professor Keyser has said: "Choices differ but some choice of principles we must make . . . and when we have made it, we are at once bound by a destiny of consequences beyond the power of passion or will to control or modify; another choice of principles is but the election of another destiny." The disturbing and dangerous side of the question is that the great majority of mankind are unaware of the silent doctrines which govern them. They take labels, creations of their own rational will for objects, and objects for events as true constituents of nature, and they fight and die for them.

We have come to the point where mathematics and our daily language meet. They both of them operate with concepts which, in the last analysis, are disguised definitions, generalizations. assumptions. In this respect the concepts "a cosine" and "a man" are identical, neither "a cosine" nor "a man" physically exists (John Smith, or Bill Brown exists, but not "a man"). A cosine and a man are both conceptual constructions. The "a cosine" is defined consciously and precisely; the other term "a

man'' has *no* scientific definition; we are still in the caveman stage of confusion about this most important of all terms. Mathematicians are conscious of what they do; others are less so. That is why mathematical achievements stand better than any others.

Let me point to a fact which seems to me to be extremely important, and which I shall call the ''Physiological point of view of mathematics.'' We have seen that man has a great freedom in building up his abstractions. It happens that in mathematics the external universe has imposed the generalizations upon man, whereas, in the other disciplines, man has imposed his fancy upon external nature.

Let me explain a little. Modern mathematics deals formally with what can be said about anything or any property. Here it may be explained why mathematics has this exclusive position among the sciences. It must be emphasized that it was not some special genius of the mathematician as such, that was responsible for it. With the coming into existence of the rational being—man—rational activity began spontaneously (no matter how slowly) and this rational activity manifested itself in every line of human endeavor—no matter how slight such activity was. Today we know that we humans can know nothing but abstractions. The process of constructing abstractions is quite arbitrary. Since man began he plunged into this process of constructing arbitrary abstractions—it was the very nature of his being to do so.

Obviously, in the beginning, he did not know anything about the universe or himself; he went ahead spontaneously. It is no wonder that some of his abstractions were false to facts, that some of them were devoid of meaning, and hence neither true nor false but strictly meaningless, and that some of them were correct. In this endless spontaneous process of constructing abstractions he started from that which was the nearest to him—namely his own feelings—and ignorantly attributed his human faculties to all the universe around him. He did not realize that he—man—was the latest product in the universe; he reversed the order and anthropomorphized all around him. He objectified his labels, mistook them for events, and became an ''absolutist.'' He did not realize, and this is true even today in most cases, that by doing so he was building up a logic and a language

ill fitted to deal with the actual universe, with life, including man; and that by doing so he was building for himself mental impasses. In a few instances good luck was with him; he made a few abstractions which were at once the easiest to handle and were correct; that is, abstractions corresponding to the actual facts in this actual universe.

These were numbers.

Let us see what was and is the significance of numbers. Any one may see that there are actual differences between such groups as * or as *, *, or as *, *, *, whatever the group was composed of, be it stones, figs, or snakes. And man could not miss for long the peculiar similarity between such a class * * of stones or such a class * * of snakes, etc., and here happened a fact of crucial signifance for the future of man. He named those different classes by definite names; good luck saved mankind from his ignorant speculations; he called the class of all such classes as * "one," the class of all such classes as * * "two," * * * "three," etc., and number was born.

Here as everywhere else "le premier pas quite coûte"; number being created the rest followed as a comparatively easy task. Man could not long fail to see that if such a class * is joined to such a class *, he gets such a class * *, but the other day he had called such classes "one" and "two", and so he concluded that "one and one makes two"—mathematics was born—exact knowledge had begun.

Good luck combined with his human faculties thus helped him to discover one of the eternal truths.

The creation of number was the most reasonable, the first truly scientific act done by man; in mathematics this reasonable being produced a perfect abstraction, the first perfect instrument for training his brain, his nerve currents, in the ideal way befitting the actual universe (not a fiction) and himself as a part of the whole. Now it is easy to understand, from this physiological point of view, why mathematics has developed so soundly. The opposite can be said about the other disciplines. In the main they started with fictions, and even today the fictions persist, and bring havoc in the life of man.

Mathematics alone started aright!

To professional mathematicians all that I have said here may appear as platitudes hardly worth mentioning. I have taken

the liberty of repeating them to show that this system of doctrinal functions, of pure thought, which we call pure mathematics (Keyser) has a direct and most vital application to all the other problems of man.

In order to deal rationally with any object, no matter what, though it is not always possible, it is always desirable, to have an analytical definition of the object. In this case where the object is man, the importance of such a definition is absolutely indispensible for the obvious reason that the results of all our thinking about man depend upon what we humans think man is.

Without an analytical, sharp, and precise definition, no demonstration is possible. How can we hope to establish anything whatever about a term if we do not take into account its meaning, is conceptual content? Now the content is given by the definition and by it alone. No definition, no demonstration.

At the very outset of our journey we find a fact so astonishing—so shocking—that it takes some effort to admit the shameful truth. Man deals with man without a scientific definition of man. Some day treatises will be written on this subject alone and in such treatises the responsibility will be traced for this calamitous omission in the intellectual life of humanity.

A definition of man is, of course, the first concern of human engineering. How shall we define our object, man? We are told by the naturalists that an organism must be treated as a whole—that sounds impressive—but they have not told us how to do it. It seems that the traditional subject-predicate logic leads automatically toward elementalism, and that this organism-as-a-whole theory will forever remain pia desideria as long as we use the old logic. Yet this concept of the "organism-as-a-whole" is extremely important for us, particularly in the dealing with man (see Manhood of Humanity), and all experimental evidence seems to prove that it is correct. We are told on the other hand, that the organism is too complicated to be treated mathematically. It seems to me that these two statements are incompatible. Of course, it is true that, if we pursue the elementalist's point of view, then the organism is too complicated; but, if it is a "whole," then, if a proper generalization is found, the "organism-as-a-whole" *could* be treated mathematically because we could deal with this one generalization.

By definition I do not mean a nominal definition which is merely the fixing of a name, a label to an object, but that analytical definition which will enable us to make the greatest number of general and significant assertions. Let us see how we could define man. Man, among all living beings, is the only one which has a chin; this characteristic is unique. Also he is the only mammal having no tail.[1] We could, if we chose, define man as a "chinful" or "tailless" mammal; these definitions would comply with the minor conditions for a real definition but they would not comply with the major condition, without which a definition is not a real definition, namely, it would not give important logical results. These examples alone show that we could define man in a great many ways, yet the definitions would be practically worthless or fruitless. It is simpler by far to find out by reflection, what are the terms in which an ideal definition of man should be made, and a definition which would, if possible, give us the "organism as a whole."

To find such definitions is not difficult, but what is extremely difficult is to have the moral courage to admit the sad fact, that, in spite of all advancement of science, man—the creator of science—deals with man on the old mythological base.

If we go back to our schoolbooks, we will find in an old edition of the "Elements of Logic" by Jevons-Hill, published by the American Book Co., in 1883, just 40 years ago, that: "It is necessary to distinguish carefully the purely logical use of the terms genus and species from their *peculiar* use in natural history. . . . If we accept Darwin's theory of the origin of species, this definition of species becomes entirely illusory, since different genera and species must have, according to this theory, descended from common parents. The species then denotes a *merely arbitrary* amount of resemblance which naturalists choose to fix upon, and which it is not possible to define more exactly. This use of the term, then, *has no connection whatever with the logical use* . . . " (page 230-231, italics are mine). Surely blind prejudices are still active, and they are doing their work thoroughly, because, as yet, the need for a scientific definition of man is still ignored.

To perform our task we will have to observe, and think, and this little old book of logic at once gives us the valuable advice

[1]Since the delivery of this lecture the author has seen pictures of a savage tribe with tails. This fact, be it a fact, does not alter the argument.

that: "Nothing is more important in observation and experiment than to be *uninfluenced* by any prejudice or theory" (page 207, italics are mine).

Just that is the first great obstacle in our path, for since our birth, we have been fed with mythological, fundamentally false ideas about the distinctive nature of man. The struggle to overcome this will be hard, as all possible odds are against us and a free independent logical issue. Once this clearing of the way is accomplished, and I know too well how difficult it is to free oneself from prejudices, nothing of importance stands in the way.

The ideal definition for man would be a definition in the same terms in which, in the exact sciences, we have attempted the formulation of the universe around us. The benefit of such a definition would be, that it would be in familiar terms and would keep man logically inside of the universe, as an actual part of it.

Observing living beings, we find that the plants bind solar energy into chemical energy, and so we may define plants as the energy or chemistry-binding class of life. The animals have an added mobility in space—they are the space-binding class of life. Humans differ from animals in that each generation does not begin where their respective ancestors began; they have the faculty to begin where their ancestors left off; they benefit by and accumulate the experiences of all the past, add to it and transmit it to the future. Man and man alone is active in a peculiar way in what we call time—so we must define man as a time-binding class of life.

The above definitions are self evident when stated. Here we must at once make clear that we have to use a static language to cover the dynamic march of events. The classes of life overlap but so do the physical, "matter," "space," "time" overlap. So in our definition we are true to facts. Matter, space and time which do not overlap are abstractions and abstractions only, and I use them *as such*.

It is easy to see that this definition of man is unique. Beyond doubt animals did not produce civilization—man did, and he was able to do so because, and only because, of his capacity to bind time. Here we get for the first time, the logic of the "organism as a whole" as applied to man and the affairs of man.

To produce this long desired logic, a new concept, a new generalization was needed.

Heat is measured not by heat but by the effect of heat; in the same way, by this new generalization, a mathematical treatment of man becomes possible, by the analysis of man's activities. This leads to the exponential function of time, "PR^T" as given in the Manhood of Humanity.

Now what of the logical fertility of this definition? The consequences of it far surpass our most sanguine dreams, the details of which are to be found in my book I mentioned before. I will mention here only a few.

The law of the survival of the fittest remains true, but true in the proper type or dimension; survival of the fittest in space is a natural law for space-binders. Physics tells us that two bodies cannot occupy the same space at the same time, and, therefore, the survival of the fittest in space—the obvious law of animals—means brutal fight where the strongest, most ruthless, survives. With the time-binder the same law takes on an entirely different aspect. To be a natural law for time-binders it must be the survival of the fittest in time. Who indeed "survives in time"? The strongest or the best? Here at once we come to a foundation on which scientific ethics can be built.

A short inquiry will easily reveal that most of our civilization hitherto has been built upon the generalizations taken from animal life. This man-made civilization was an "animal" civilization because of the fundamentally wrong ideas man had of himself.

This definitioin also complies with the mathematical theory of logical types or, as I prefer to call it, the theory of dimensionality. It is obvious that animal and man are different types, they are of different dimensionality, as different factors enter which make them distinct. The realization of this makes it obvious that no rule, no generalization taken from animal life, will apply to man any more than rules of surfaces will apply to volumes. If we confuse our types or mix our dimensions in reasoning about man, his structures (called civilization, in this case) must collapse every little while; just as a bridge built on false formulas, would collapse. All the tragic history of mankind proves that this conculsion is true. Man is not a mixture of beast and angel, but man is man, and must learn to think of himself as such.

Professor Keyser in his "Mathematical Philosophy" has done me the honor to devote a chapter to the new concept of man. I am frank to say that it is the best analysis of the concept in existence. He made here an important addition, namely, that for animals it matters what animal is; for man it matters not only what man is, but even more what man thinks man is. *One* factor for animals, *two* factors for man.

These simple but undeniable observations at once prove that the fashionable school of behaviorists is perfectly scientific in respect to all creatures below man. In respect to man, their doctrine appears fallacious. Their doctrine deals only and exclusively with what something *is*, how it behaves in the animal dimension; but it cannot deal in the same fashion, without grave error, with something in which two factors enter, namely, what this something is and what it thinks it is. It is the same as applying the rules of surfaces to volumes; this would be poor mathematics; all our bridges would collapse in the same way our social structures recurrently collapsed, because built upon a conception of human nature.

It does not really matter much if the definitions as given here will survive for long, what matters and matters much, is the fact that we see clearly our neglect and the new and fertile fields now open for inquiry. The theory of time-binding is the study of the "behavior" of man, and man alone, but in its proper dimension, true to facts and free from logical confusion.

The old civilization is crumbling. The new will require a complete revision of old fallacies and prejudices, and most probably mathematicians, who are today the best logically trained men, will be very active and productive in this coming reconstruction of science and life. A thorough going scientific revision will lead to a complete reversal of many traditional beliefs. It will be found that the belief in the existence of non-existents such as, metaphysical "infinitessimals," metaphysical "infinite," "absolute space," "absolute time," is very wide spread; indeed it embraces practically the whole of humanity. This has been taught to us since our birth; it is even taught in some schools and universities today by such expressions as "matter is that which occupies space," and similar fallacies which fatalistically lead by the law of logical fate, which applies to all, educated or non-educated, civilized or non-civilized, to a world-

conception, contrary to human nature. Such conceptions are deadly, they lead to mental impasses, making man feel hopeless and helpless in a hostile and strange universe; he rebels and this leads him to mystical and mythological delusion, which also fail him. This feeling of hostility all around him transforms him into a hostile being, and the antique proverb: "Homo homini lupus" is too often a bitter, yet entirely logical consequence of the silent or conscious assumptions of the truth of fundamental fallacies.

Yet the actual universe is *not* hostile; it is at most, indifferent. The vicious fictions, the abuse of his power to assume, to abstract, to generalize and invent non-existents has vitiated the whole outlook of man, in all fields. Man saw that animals fight and he imposed upon himself "fighting" as the "manly" art, and blinded by his prejudices and vicious logic he did not stop to think that cooperation—which has been and is now artificially hampered—is the basic law of a *human, time-binding,* rational class of life.

Any inquiry into the above mentioned problems and their mathematical solutions will disclose that no branch of human knowledge has ever contributed more to humanity than the mathematical inquiry into mathematical foundations. The psychological transformation will be complete. Man will understand himself. Needless to say that the semi-sciences will be transformed into sciences.

A new school of history will arise which will show to mankind what disasters the wrong conception of man by man has wrought to mankind. Philosophers will compile charts of "logical destiny," showing what consequences one concept, one abstraction, one generalization have brought to us. This probably will bring mankind to its senses, and this will probably start the true reconstruction of science and life.

Allow me to summarize my lecture and try to justify its title. Mathematical discoveries of the last few decades, culminating lately in the works of Whitehead, Russell, Keyser and Einstein, have made us conscious of the power of rigorous thought and have also disclosed the inner structure and working of this subtle instrument called human thought. They have proved and it has been ultimately formulated by Keyser, that human freedom is not absolute; that we are governed by logical fate. We are free

to select our assumptions; if we select false assumptions, disaster follow. But to exercise this freedom, man must first know that he is thus free; otherwise he will continue to accept false assumptions, the old language, etc., as final "innate ideas," etc., without realizing that the moment he does so, he renounces the freedom he has, and becomes the slave of logical fate of his creeds.

This also explains why mankind is divided into so many fighting factions. We are not conscious of the silent, often false assumptions which underlie our language and actions, how do we expect to prove anything to the satisfaction of all if we do not possess a scientific definition of man? As was said before: No definition, no demonstration, no demonstration, no agreement possible.

A diagram may help the visualization of these few ideas.

old — old
false — Logical Destiny — false
premises $A >\longrightarrow B$ theories
assumptions — creeds
postulates — In — beliefs, systems
con
sis
ten
cy
E

new, truer — new, truer
premises $C >\longrightarrow D$ theories
assumptions — Logical Destiny — systems, etc.
postulates

If we start with A, as most of us do, we can *not* reach D and *convince* all, because inconsistencies E arise which prevent the universal acceptance of some high-sounding but logically unsound doctrines. If we want to reach D, the new and truer theory, we *must* start with new and more fundamental, truer premises. In order to know which are truer we *must first investigate them*, without being shy about it.

No doubt mathematicians, and those who have mathematical training are the best fitted for this work. There are signs that this work has already been started, and indeed, nothing could be more important for the future of man.

MECHANICS

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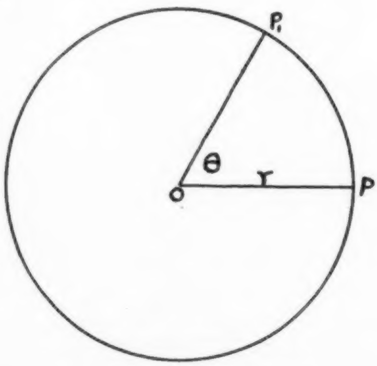
ANGULAR MOTION. APPLICATION.

If any one of us should enter a machine shop we would find a good many cases of rotation about a fixed axes. As we look around us we see revolving wheels, cylinders, shafts, etc.

Suppose some force that is applied to a shaft is causing it to revolve 75 times in a minute. If a wheel is fast to this shaft it would also revolve 75 times in a minute. It is common practice to speak of angular speed or speed of turning as so many revolutions per unit of time, usually per minute or second.

Another unit sometimes used for measuring angular speed is the angle called the radian. This is the angle at the center of a circle which is subtended by an arc equal in length to the radius. In our work we will express the angular speed as so many radians per minute or second instead of so many revolutions per minute or second.



$$\text{arc } P_1P = \text{radius} = r$$

Hence, angle $\theta = 1$ radian

$$\text{for } \theta = \frac{\text{arc } P_1P}{r} = 1$$

If we regard the whole circumference as an arc of length $2\pi r$, we have

$$\theta \text{ in radians} = \frac{\text{circumference}}{r} = \frac{2\pi r}{r} = 2\pi.$$

Hence, $360^\circ = 2\pi$ radians
or 1 radian $= 57^\circ\ 17'\ 42''$

Problems

1. Reduce to radians: 45°; 100°; 900°; 4 revolutions. Ans.: 0.7845. 1.746. 15.71. 25.13.
2. Reduce to degrees: 2 radians; π radians.

The Mechanics of Angular Motion

Comparison of Symbols in Linear and Angular Motion.

	linear	angular
acceleration	a	α
velocity	v	w
distance	S	θ

Now the first question: Is there any relation between linear and angular motion?

Let a wheel on this shaft turn through 3 radians every second. Now to obtain the angle turned out during 5 seconds we multiply 5 by 3 and get an angle of 15 radians. Here $\theta = 5 \cdot 3 = 15$ radians.

If it turned out w radians a second for t seconds, we have

$$\theta = w.t$$

$$\text{or } w = \frac{\theta}{t}$$

If in figure I point P moves to point P_1 with a linear velocity of v ft./sec. we have arc $PP_1 = v \cdot t$.

$$\text{Hence, } \theta = \frac{\text{arc } PP_1}{r} = \frac{v \cdot t}{r}$$

$$\text{but } wt = \frac{v \cdot t}{r} \text{ for } \theta = w.t$$

$$\text{or } w = \frac{v}{r}$$

Therefore, if we know the linear velocity of a point on the rim of a wheel all we have to do is to divide it by the radius to get the angular velocity in radians.

In angular motion the time rate of change of angular velocity is angular acceleration.

$$\text{or } \alpha = \frac{w}{t}$$

$$\text{for } w \text{ place } \frac{v}{r}$$

$$\text{whence } \alpha = \frac{v}{rt}$$

$$\text{but } \frac{v}{t} = a$$

$$\text{Therefore, } \alpha = \frac{a}{r}$$

Now let us look for a relation between θ and S.

$$\theta = \frac{\text{arc } PP_1}{r} = \frac{S}{r}$$

Now let us sum up our results.

$$w = v/r \qquad \alpha = a/r \qquad \theta = S/r$$

$$\text{or } v = wr \qquad a = \alpha r \qquad S = \theta r$$

If we place the above results for v, a and S in the following formulas,

$$v = v_0 + at$$
$$S = v_0 t + \tfrac{1}{2}at^2$$
$$v^2 = v_0^2 + 2aS$$

we have the corresponding formulas for angular motion.

$$w = w_0 + \alpha t$$
$$\theta = w_0 t + \tfrac{1}{2}\alpha t^2$$
$$w^2 = w_0^2 + 2\alpha\theta$$

NOTE. These formulas could have been derived by the same methods that we used under linear motion.

PROBLEMS

1. If a flywheel of 10 ft. diameter makes 30 revolutions per minute, what is its angular velocity, and what is the linear velocity of a point on its rim?

Solution:

30 rev./min. $= \frac{1}{2}$ rev./sec. $= \pi$ radians/sec. $= w$.

Answer:

NOTE

$$v = r \cdot w \left| \begin{array}{l} r = 5 \text{ ft.} \\ w = \pi \text{ radians} = 3.1416 \text{ radians} \end{array} \right.$$

$$v = 5 \cdot 3.1416 = 15.7080 \text{ ft./sec.} \quad \text{Ans.}$$

2. A particle describes a circle radius 5 feet with a uniform linear velocity of 8 ft./sec. Find the angular velocity. Ans. 8/5 radians per sec.

3. The lengths of the hour, minute and second hands of a watch are .48, .8 and .24 inches respectively. Find the ratios of the angular velocities; also of the linear velocities of the ends of the hands. Ans.: Angular 1:12:720; linear 1:20:360.

4. A flywheel rotates initially 3 times per second, and after twenty seconds is making 140 revolutions per minute. How many revolutions will the wheel make, and what time will elapse before stopping, if the retardation is uniform? Ans.: 135 revolutions; 90 sec.

5. A flywheel is brought from rest up to a speed of 60 r.p.m. in $\frac{1}{2}$ minute. What is the angular acceleration, and the number of revolutions required? What is the velocity at the end of 15 seconds? Note: r.p.m. = revolutions per minute. Ans. 0.2094 radians/sec^2. 15 revolutions. 3.14 radians/sec.

6. A pulley which is rotating at 120 r.p.m. comes to rest under the action of friction in 3 minutes. What is the angular acceleration and the total number of revolutions made? Ans.: 0.0698 radians/sec^2. 180 revolutions.

7. A wheel is running at a uniform speed of 32 turns a second when a resistance begins to retard its motion uniformly at a rate of 8 radians per second. (a) How many turns will it make before stopping? (b) In what time is it brought to rest? Ans.: (a) 402. (b) 25.1 sec.

8. A wheel of 6 ft. diameter is making 50 r.p.m. when thrown out of gear. If it comes to rest in four minutes, find (a) the angular retardation; (b) the linear velocity of a point on the rim at the beginning of the retarded motion; (c) the same after two minutes. Ans.: 0.00218 radians/sec^2. (b) 15.7 ft./sec. (c) 7.86 ft./sec.

VARIETIES OF MINUS SIGNS

By PROFESSOR FLORIAN CAJORI
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One of the curiosities in the history of mathematical notations is the fact that notwithstanding the extreme simplicity and convenience of the symbol — to indicate subtraction, a more complicated symbol of subtraction ∸ should have been proposed and been able to maintain itself with a considerable group of writers, during a period of four hundred years.

It is well known that the first appearance in print of the symbols + and — for plus and minus is found in Johann Widman's arithmetic of 1489, published at Leipsig. The sign — is one of the very simplest conceivable; therefore it is surprising that a modification of it should ever have been suggested.

Probably the printed signs have ancestors in hand-written documents, but the line of descent is usually difficult to trace with certainty. The following quotation suggests a clue[1]: "In the westgothic writing before the ninth century one finds, as also Paoli remarks, that a short line has a dot placed above it ∸, to indicate *m*, in order to distinguish this mark from the simple line — which signifies a contraction or the letter *n*. But from the ninth century down, this same westgothic script always contains the dot over the line even when it is intended as a general mark."

In print the writer has found the sign ∸ for *minus* only once. It occurs in the 1535 edition of the Rechenbüchlin of Grammateus.[2] He says: "Vnd mañ brauchet solche zeichen als + ist mehr / vnd ∸ / minder." Strange to say, this minus sign does not occur in the first edition (1518) of that book. The corresponding passage of the earlier edition reads: "Vnd man braucht solche zaichen als + ist vnnd / — mynder." Nor does Grammateus use ∸ in other parts of the 1535 edition; in his mathematical operations the minus sign is always —.

The use of the dash and two dots, thus ÷, for minus, has been found by J. W. L. Glaisher for the first time in 1525, in

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[1] Adriano Cappelli, *Levicon Abbreviaturam,* Leipzig, 1901, p. XX.

[2] Henricus Grammateus, *Eyn new Künstlich behend and gewiss Rechenbüchlin,* 1535 (first edition 1518). For a facsimile page of the 1535 edition, see D. E. Smith, *Rara arithmetica,* 1908, p. 125.

an arithmetic of Adam Riese[1], who explains: "Sagenn sie der warheit zuuil so bezeychenn sie mit dem zeychen + plus wu aber zu wenigk so beschreib sie mit dem zeychen ÷ minus genant."[2]

No reason is given for the change from — to ÷. Nor did Riese use ÷ to the exclusion of —. He uses ÷ in his algebra, "Die Coss," of 1524, which he did not publish, but which was printed[3] in 1892, and also in his arithmetic, published in Leipzig in 1550. Apparently, he used — more frequently than ÷.

Probably the reason for using ÷ to designate — lay in the fact that — was assigned more than one signification. In Widman's arithmetic — was used for subtraction or minus, also for separating terms in proportion,[4] and for connecting each amount of an article (wool, for instance) with the cost per pound.[5] The symbol — was also used as a rhetorical symbol or dash in the same manner as it is used at the present time. No doubt, the underlying motive in introducing ÷ in place of — was the avoidance of confusion. This explanation receives support from the German astronomer Regiomontanus[6] (1436-1476) who, in his correspondence with the court astronomer at Ferrara, Giovanni Bianchini, used — as a sign of equality; hence used for subtraction a different symbol, namely $\overline{ig}$, with him 1$\overline{ig}$ r^e meant $1 - \times$.

Eleven years later, in 1546, Gall Spenlin of Ulm had published at Augsburg his "Arithmetica Künstlicher Rechnung,"[7] in which he uses ÷, saying: "bedeut das zaichen + züuil, und das ÷ zü wenig." Riese and Splenlin are the only arithmetical authors preceding the middle of the sixteenth century whom Glaisher mentions as using ÷ for subtraction or minus.[8]

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1 Adam Riese *Rechenung auff der linihen vnd federn in zal, masz, vnd gewicht,* Erfurt, 1525 (first edition 1522).

2 This quotation is taken from J. W. L. Glaisher's article, "On the Early History of the Signs + and — and on the early German arithmeticians," in *The Messenger of Mathematics,* Vol. 51, 1921, p. 36.

3 See Bruno Berlet, *Adam Riese,* Leipzig, Frankfurt A/M, 1892.

4 J. W. L. Glaisher, *loc. cit.* p. 15.

5 J. W. L. Glaisher, *loc. cit.* p. 16

6 M. Curtze in *Abhandlungen zur Geschichte der Mathematischen Wissenschaften,* Vol. 12, 1902, p. 234.

7 J. W. L. Glaisher, *loc. cit.,* p. 43.

8 J. W. L. Glaisher, *loc. cit.,* Vol. 51, p. 1-148.

With the beginning of the seventeenth century ÷ for minus appears more frequently, but, as far as we have been able to ascertain, only in German, Swiss, and Dutch books. A Swiss writer, W. Schey, in 1600[1] and in 1602 uses both ÷ and ÷ for minus. He writes 9 + 9, 5 ÷ 12, 6 ÷ 28, where the first number signifies the weight in *centner* and the second indicates the excess or deficiency of the respective casks in *pounds*. In another place Schey writes "9 fl. ÷ 1 ort," which means 9 florins less 1 ort or quart. In 1601 Nicolaus Reymers,[2] an astronomer and mathematician, uses regularly ÷ for minus or subtraction; he writes

XXVIII	XII	X	VI	III	I	0
1 gr.	65532	+ 18	÷ 30	÷ 18	+ 12	÷ 8

for $x^{28} = 65532x^{12} + 18x^{10} - 30x^6 - 18x^3 + 12x - 8$.

In 1608 Peter Roth of Nürnberg uses ÷ in writing[3] $3x^2 - 26\ x$. Johannes Faulhaber[4] at Ulm in Würtemberg used ÷ requently. With him the horizontal stroke was long and thick, the dots being very near to it. The year following, the symbol occurs in an arithmetic o Ludolf van Ceulen,[5] who says in one place: "Subtraheert $\sqrt{7}$ van, $\sqrt{13}$, rest $\sqrt{13}$, weynigher $\sqrt{7}$, daerom stelt $\sqrt{13}$ voren en $\sqrt{7}$ achter, met een sulck teecken ÷ tusschen beyde, vvelck teecmin beduyt, comt alsoo de begeerde rest $\sqrt{13}$ ÷ $\sqrt{7}$ —." However, in some parts of the book — is used for subtraction. Albert Girard[6] in 1629 mentions ÷ as the symbol for *minus*, but uses —. In 1617 Otto Wesellow[5] brought out in Bremen a book in which +, ÷ stand respectively for plus

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[1] Wilhelm Schey, *Arithmetica Oder die Kunst zu rechnen*, Basel. 1600, 1602. We quote from D. E. Smith, *Rara arithmetica*, 1908, p. 427, and from Matthäus Sterner, *Geschichte der Rechenkunst*, München ,and Leipzig, (1891), p. 280, 291.

[2] *Nicolai Raimari Ursi Dithmarsi . . . arithmetica analytica, vulgo Cosa*, oder *Algebra*, 1601, zu Frankfurt an der Oder. We take this quotation from C. I. Gerhardt *Geschichte der Mathematik in Deutschland*, München, 1877, p. 85.

[3] Peter Roth *Arithmetica philosophica*, 1608. We quote from P. Treutlein "Die Deutsche Coss" in *Abhandlungen zur Geschichte der Mathematik*, Vol. 2, Leipzig, 1879, p. 28, 37, 103.

[4] Johannes Faulhaber, *Numerus figuratus sive arithmetica analytica*, Ulm, 1614, p. 11, 16.

[5] Ludolf van Ceulen, *De arithmetische en geometische Fondamenten*, 1615, p. 52, 55, 56.

[6] *Invention Nouvelle en L'Algebra par Albert* Girard, Amsterdam, 1629, (no paging.)

[7] Otto Wesellow, *Flores arithmetici*, Drüdde vnde Veerde deel, Bremen, 1617, p. 523.

and minus. These signs are used in 1622 by Herman Follinus[1] in a book printed at Cologne, by Daniel van Hovcke[2] in 1628, who speaks of + as signifying "mer en ÷ min.", and in 1646 by Johann Ardüser[3] in a geometry printed at Zurich. It is interesting to observe that only thirteen years after the publication of Ardüser's book, another Swiss, J. H. Rahn, finding, perhaps, that there existed two signs for subtraction, but none for division, proceeded to use ÷ to designate division. This practice did not meet with adoption in Switzerland, but was seized upon with great avidity as the symbol for division in a far off country, England. In 1670 ÷ was used for subtraction once by Chr. Huygens[4] in the Philosophical Transactions. J. Hemelings[5] of Hanover in 1678 indicated in an example 14½ legions less 1250 men, by "14½ Legion ÷ 1250 Mann." The symbol is used in 1690 by Tobias Beutel,[6] who writes 81 ÷ 1R6561 ÷ 162. R. + 1. *zenss* to represent our $81 - \sqrt{6561 - 162x + x^2}$. J. M. Kegel[7] in 1696 explains how one can easily multiply by 41, by first multiplying by 6, then by 7, and finally subtracting the multiplicand; he writes 7 ÷ 1. In a set of seventeenth century examination questions,[8] used at Nürnberg, reference is made to cossic operations involving quantities, "durch die Signa + und ÷ connectirt."

The vitality of this redundant symbol of subtraction is shown by its continued existence during the eighteenth century. It was employed by G. H. Paricius[9] of Regensburg in 1706. It

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[1] *Algebra sive liber de rebus occultis,* Hermannus Follinus, Coloniae, 1622, p. 113, 185.

[2] Daniel van Hovcke, *Cyffer-Boeck* . . . Den tweeden Druck, Rotterdam, 1628, P. 129-133.

[3] John. Ardüser, *Geometriae theoricae et practicae. Oder von dem Feldmässen.* Zürich, 1646, folio 75.

[4] Chr. Huygens in a reply to Slusius, *Philosophical Transactions* for 1670, Vol 5, London, p. 6144.

[5] *Selbstlehrendes Rechne-Buch, . . . durch Johannem Hemelingium,* Frankfurt, 1678. Quoted from Hugo Grosse *Historische Rechenbücher des 16. and 17. Jahrhunderts,* Leipzig, 1901, p. 112.

[6] *Geometrische Gallerie* von Tobias Beuteln, Leipzig, 1690, p. 46.

[7] Johann Michael Kegel *New vermehrte artihmetica vulgaris et Practica italica,* Frankfurt am Mayn, 1696. We quote from Sterner, *op. cit.,* p. 288.

[8] Fr. Unger *Die Methodik der Praktischen Arithmetik in Historischer Entwickelung,* Leipzig, 1888, p. 30.

[9] Georg Heinrich Paricius *Praxis Arithmetices,* 1706. We quote from Sterner *op. cit., p.* 349.

was used very frequently as the symbol for subtraction and minus in the earliest mathematical journal ever published, the *Maandelykse Mathematische Liefhebbery,* brought out in Amsterdam, 1754-1769. It is found in a Dutch arithmetic by Bartjens[1] which passed through many editions. The vitality of the symbol is displayed still further by its regular appearance in a book by G. van Steyn[2] (1768), who, however, uses — in 1778.[3] P. Halcke[4] in 1768 states, " ÷ of — het teken van *substractio, minus* of min." but uses — nearly every where. The ÷ occurs in 1788 in a Leipsig magazine,[5] in a Dresden work by C. C. Illing[6] in 1793, in a Berlin text of 1817 by F. Schmeisser.[7] In a part of G. S. Klügel's[8] mathematical dictionary, published in 1831, it is stated that ÷ is used as a symbol for division, "but in German arithmetics is employed also to designate subtraction." A later use of it for minus, that we have noticed, is in a Norwegian arithmetic[9] of 1869. In fact, in Scandinavian countries the sign ÷ for minus is found occasionally in the twentieth century. For instance, in a Danish scientific publication of the year 1915, a chemist expresses a range of temperature in the words "fra + 18°C. til ÷ 18°C."[10] The difference in the dates that have been given, and the distances between the places of publication, make it certain that this symbol ÷ for minus had a much wider adoption in Germany, Switzerland, Holland and

[1] *De vernieuwde Cyffering van Mr. Willem Bartjens, . . . vermerdert-ende verbetert, door Mr. Jan van Dam . . . en van alle voorgaande Fauten gezuyvert door Klaas Bosch,* Amsterdam, 1771, p. 174-177.

[2] Gerard van Steyn *Liefherbbery der Reekenkonst,* eerste deel, Amsterdam, 1768, p. 3, 11, etc.

[3] G. van Steyn, *op. cit.,* 2e Deels 2e Stuk, 1778, p. 16.

[4] *Mathematisch Zinnen-Confect* . . . door Paul Halcken . . . Uyt het Hoogduytsch vertaald . . . dor Jacob Oostwoud. Tweede Druk, Te Purmerende, 1768, p. 5.

[5] J. A. Kritter in the *Leipziger Magazin für reine und angewandte Mathematik,* herausgegeben von J. Bernoulli und C. F. Hindenburg, 1788, p. 147-161.

[6] Carl Christian Illing. *Arithmetisches Handbuch für Lehrer in den Schulen,* Dresden, 1793, p. 11, 132.

[7] Friedrich Schmeisser, *Lehrbuch der reinen Mathesis,* 1. Theil, Berlin, 1817, p. 45, 201.

[8] G. S. Klügel, *Mathematisches Wörterbuch,* Art. "Zeichen."

[9] G. C. Krogh, *Regnebog for Begyndere,* Bergen, 1869, p. 15.

[10] Johannes Boye Petersen, in *Kgl. Danske Videnok. Selskabs. Skrifter, Nat. og. Math. Raekke* X, 5, Kopenhagen, 1915, p. 330. See also pages 221, 223, 226, 230, 238.

Scandinavia than the number of our citations would indicate. But its use seems to have been confined to Teutonic peoples.

Several writers on mathematical history have incidentally called attention to one or two authors who used the symbol $\div$ for minus, but none of the historians revealed even a suspicion that this symbol had an almost continuous history extending over four centuries. This article is the first to establish the existence of this long career.

It is well known that the Italian Pacioli in 1494 and the Italian mathematicians of the sixteenth century designated *plus* by $\bar{p}$ and *minus* by $\bar{m}$. The $+$ and $-$ invaded Italy in 1608 when the German mathematician, Clavius, then resident in Rome, brought out there his algebra, containing these symbols.

The Italian $\bar{p}$ and $\bar{m}$ are encountered in some French and Spanish texts, but I know of only one English author[1] using them, namely, the physician and mystic, Robert Fludd, whose numerous writings were nearly all published on the Continent. Fludd uses $\frac{\text{P}}{\text{l}}$ and $\frac{\text{M}}{\text{m}}$ for plus and minus.

Sometimes the minus sign — appears broken up into two or three successive dashes or dots. In a book of 1610 and again of 1615, by Ludolph van Ceulen,[2] the minus sign occasionally takes the form - - . Rich. Balam[3] (1653) uses three dots and says "3 - - - 7, 3 from 7"; he writes an arithmetical proportion in this manner: "2 - - - 4 = 3 - - - 5." Two or three dots are used in the *Journal des Sçavans*, in René Descartes' *Geometrie* (1637), in the writings of M. Mersenne[4] (1644), and in many other seventeenth century books.

From these observations it is evident that in the sixteenth and seventeenth centuries the forms of type for minus were not yet standardized. For this reason, several varieties were sometimes used in one and the same volume and even on one and the same page. Similar remarks apply to the $+$ sign and to other mathematical symbols.

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[1] See C. Henry in *Revue archeologique*, N. S. Vol. 37, Paris, 1879, p. 329, who quotes from Fludd's *Utriusque cosmi . . . Historia*, Oppenheim, 1617.

[2] Ludolphi à Ceulen *circvlo et adscriptis liber . . . Omnia e vernacülo Latina fecit et annotationibus illustravit Willebrordus Snellius*, Leyden, 1610, p. 128.

[3] Rich. Balam, *Algebra*, London, 1653, p. 5.

[4] Marin Mersenne, *Cogitata Physico-Mathematica*, Paris, 1644, Praefatio generalis, De Rationibus atque Proportionibus, XII, XIII.

The contents of this paper emphasizes the difficulty experienced, even in ordinary arithmetic and algebra, in reaching a common world language. Centuries pass by before any marked step toward uniformity is made. It appears indeed as if blind chance were an uncertain guide to lead us away from the babel of languages. The only hope for rapid approach of uniformity in mathematical symbolism lies in international cooperation through representative committees.

CORRELATION OF THE MATHEMATICAL SUBJECTS DEVELOPS MATHEMATICAL POWER

By CHARLES A. STONE
University High School, University of Chicago

The problem below was recently submitted to the pupils of the University High School. It was published in the University High School Daily. Interest in the problem was aroused by the fact that it previously had been inserted for two weeks in the daily of a technical college in the city. It was the talk of the college for about two weeks but only four incorrect solutions were turned in, in spite of the fact that it was specifically stated that the problem could be solved by simple arithmetic. This added to the interest of the project.

The Problem

Candle A is one inch longer than candle B. Candle B is lighted at 2:30 P. M. Saturday and candle A is lighted at 6:00 P. M. Saturday. At 5 A. M. Sunday morning they burn to a level. Candle A burns out at 7:20 o'clock Sunday evening and candle B burns out at 2:30 o'clock Monday morning. Find the lengths of the candles.

The time limit for handing in the solutions was one week, and a prize was offered for the first correct original solution. Twenty-four solutions were offered and eighteen of these were correct. The boy that won the prize for briging in the first correct solution handed in his solution one and one-half hours after the daily came out. Various types of solutions were presented, the first being as follows:

Solution 1.

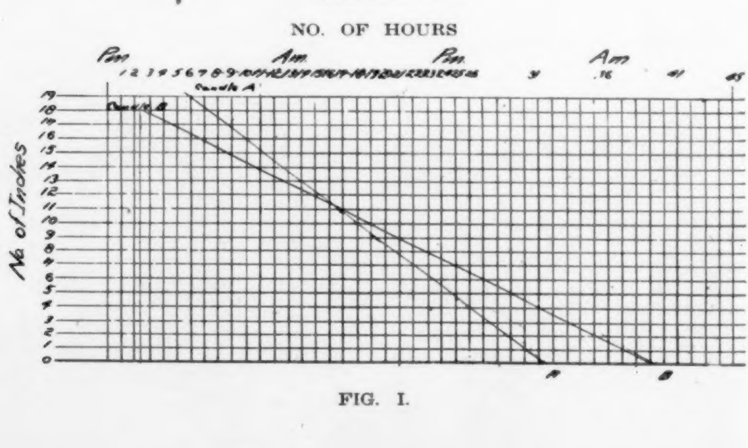

FIG. I.

The above solution was presented by a pupil of the sub-freshman class consisting of seventh grade pupils. The class had been studying about the presentation of numerical facts by means of the table, the graph, and the formula, stress being put upon the graph. Thus the graphical solution of the problem by several pupils of this class demonstrated that the graph serves the student even in the seventh grade as an instrument of mathematics. The solution indeed represents true genius and shows what young boys and girls can do in mathematics if they are but given the opportunity to learn something besides arithmetic. Even to the writer this method of solution has not occurred although he has known of this problem for several years. He has worked it a great many times, but without even the slightest suspicion that it could be solved graphically.

Solution II

Let $x =$ length of candle B
$x + 1 =$ length of candle A
Candle A burns $25\frac{1}{3}$ hours
Candle B burns 36 hours
After level is reached candle A burns $14\frac{1}{3}$ hours
After level is reached candle B burns $21\frac{1}{2}$ hours

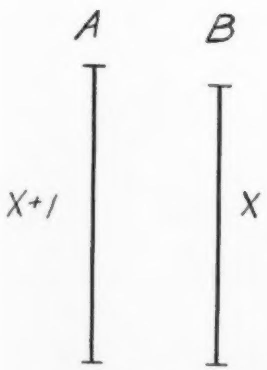

FIG II.

$$r \times t = d$$
$$\text{or } r = d/t$$
$$\text{rate of A} = \frac{x+1}{25\frac{1}{3}}$$
$$\text{rate of B} = \frac{x}{36}$$
$$\frac{x+1}{25\frac{1}{3}} \cdot 14\tfrac{1}{3} = \frac{x}{36} \cdot 21\tfrac{1}{2}$$
because the distances are equal
$$\frac{43x + 43}{76} = \frac{43x}{72}$$
$$72(43x + 43) = 76 \cdot 43x$$
$$3096x + 3096 = 3268x$$
$$172x = 3096$$
$x = 18$ length of candle B
$x + 1 = 19$ length of candle A

The boy who handed in this solution recognized the fact that the problem was a motion problem, and his work shows that

he has mastered the technique of solving this type of problem. His ability to manipulate fractions and fractional equations is also shown.

Solution III

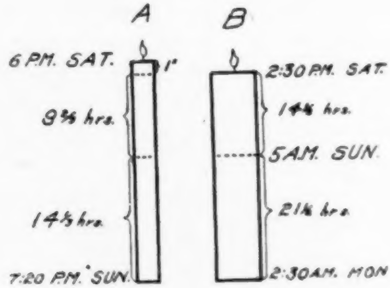

FIG III.

From the diagram it can be seen that it takes candle A $14\frac{1}{3}$ hours to burn out from the time they are at the same level at 5:00 A. M. Sunday, and it takes candle B $21\frac{1}{2}$ hours to burn out from the time they are at the same level.

Thus the relation between the rates of burning of the candles is $14\frac{1}{3} \div 21\frac{1}{2} = \frac{2}{3}$; or it takes candle A $\frac{2}{3}$ as long to burn through a given distance as it does candle B. From 2:30 P. M. Saturday to 5:00 A. M. Sunday candle B burns $14\frac{1}{2}$ hours through the same distance A burns $\frac{2}{3} \times 14\frac{1}{2}$ hours, or $9\frac{2}{3}$ hours. From 6:00 P. M. Saturday to 5:00 A. M. Sunday candle A burns 11 hours. Thus for one inch the difference between 11 and $9\frac{2}{3}$ or $1\frac{1}{3}$ hours are consumed. Total time of burning candle A is $25\frac{1}{3}$ hours.

$$\therefore \text{Length of A} = 25\tfrac{1}{3} \div 1\tfrac{1}{3} = 19 \text{ inches}$$
$$\therefore \text{Length of B} = 19 - 1 = 18 \text{ inches}$$

The above shows that the pupil who submitted the solution possessed the power of analysis. As in the previous solution it was seen that the problem was one of the motion type. This pupil was able to see that the comparative rates of burning could be obtained from the fact that the candles burned to the same level at 5:00 A. M. Sunday. The rate of burning per inch of candle, and the length of candles was then easily obtained.

Solution IV

Let x = length in inches of A
y = length in inches of B

(1) $x - 1 = y$ or $x - y = 1$

$$\frac{x}{\frac{76}{3}} \text{ or } \frac{3x}{76} \text{ is that part of candle A burned in 1 hour.}$$

$$\frac{y}{36} \text{ is that part of candle B burned in 1 hour.}$$

Eleven hours after A is lighted, and 14½ hours after B is lighted the candles are at the same level.

Assuming that A and B stand on the same level.

(2)
$$\frac{33x}{76} - 1 = \frac{\frac{29y}{2}}{36}$$

$$\frac{33x - 76}{76} = \frac{29y}{72}$$

(3) $1188x - 2736 = 1102y$

or $1188x - 1102y = 2736$

(4) Likewise $1188x - 1188y = 1188$

(5) $86y = 1548$

$\therefore y = 18$ inches, length of B

$\therefore x = 19$ inches, length of A

Here we have still another type of solution where simultaneous equations in two unknowns are used. Also the fact that the two candles burned to the same level at 5:00 A. M. Sunday is used as the key to the solution of the problem.

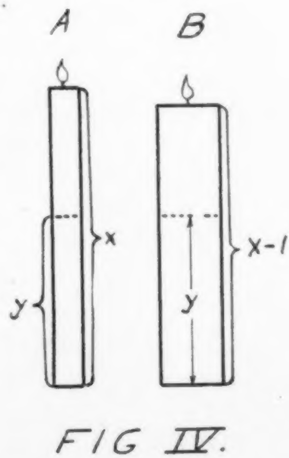

FIG IV.

Solution V

It takes eleven hours for candle A to burn the distance $x - y$, and it takes 14½ hours for candle B to burn the distance $x - 1) - y$.

It takes 14⅓ hours for candle A to the distance $(x - 1) - y$.

for candle B to burn the distance y.

$$\text{Rate of combustion of candle A} = \frac{x}{25\frac{1}{3}} \tag{1}$$

$$\text{Rate of combustion of candle B} = \frac{x-1}{36} \tag{2}$$

$$\text{Rate of combustion of candle A} = \frac{y}{14\frac{1}{3}} \tag{3}$$

$$\text{Rate of combustion of candle B} = \frac{y}{21\frac{1}{2}} \tag{4}$$

$$\therefore y = 14\tfrac{1}{3}r \tag{5}$$

$$\text{and } y = 21\tfrac{1}{2}r_1 \tag{6}$$

$$\therefore 14\tfrac{1}{3}r = 21\tfrac{1}{2}r_1 \tag{7}$$

$$x = 25\tfrac{1}{3}r,\quad x = 36r_1 + 1 \tag{8}$$

$$\therefore 25\tfrac{1}{3}r = 36r_1 + 1 \tag{9}$$

$$r = \frac{21\frac{1}{2}r_1}{14\frac{1}{3}} = \frac{36r_1 + 1}{25\frac{1}{3}} \tag{10}$$

$$\therefore \frac{21\frac{1}{2}r_1}{14\frac{1}{3}} = \frac{36r_1 + 1}{25\frac{1}{3}} \tag{11}$$

$$\frac{533\frac{2}{3}r_1}{(14\frac{1}{3})(25\frac{1}{3})} = \frac{516r_1 + 14\frac{1}{3}}{(14\frac{1}{3})(25\frac{1}{3})} \tag{12}$$

$$\text{or } 28\tfrac{2}{3}r_1 = 14\tfrac{1}{3} \tag{13}$$

$$r_1 = \tfrac{1}{2} \tag{14}$$

Substituting in (8) we get $x = 36 \times \frac{1}{2} + 1 = 19$ (15)

Result:

Candle A is 19 inches tall.
Candle B is 18 inches tall.

A glance at Solution V will impress the reader with the highly symbolical character of the method of presentation. The boy who presented the above work is not enrolled in mathematics this year but is taking courses in physics and chemistry. His training in science is reflected in this solution, and one cannot help but feel that he has the ability to apply his mathematical knowledge to situations not occurring in texts or in class.

Solution VI

Let x = length of candle B
$x + 1$ = length of candle A

It takes A 11 hours to burn to same level, and it takes B $14\frac{1}{2}$ hours to burn to same level.

Complete time of burning for A is $25\frac{1}{3}$ hours.

Complete time of burning for B is 36 hours.

$$\therefore \frac{11x + 11}{25\frac{1}{3}} = 1 + \frac{14\frac{1}{2}x}{36}$$

$$1188x + 1188 = 2736 + 1102x$$

$$86x = 1548$$

$$x = 18'' \text{ length of B}$$

$$x + 1 = 19'' \text{ length of A}$$

In this solution the statement that the candles burn to the same level at a certain time is used, and a relation between the length of the candles consumed in burning to the time is found.

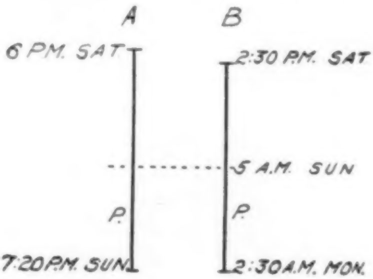

FIG. V.

Solution VII

Let A = first candle
B = second candle
V^1 = the rate of burning of candle A
V = the rate of burning of candle B
P = length of both candles at 5 A. M. Sunday

25 hours and 20 minutes = time it took candle A to burn.
36 hours = time it took candle B to burn

Let x = length of B
$x + 1$ = length of A

$$25\tfrac{1}{3}V^1 = x + 1$$

$$36V = x$$

$14\frac{1}{3}$ hours = the time it took A to burn after 5 A. M. Sunday
$21\frac{1}{2}$ hours = the time it took B to burn after 5 A. M. Sunday

$$14\tfrac{1}{3}V^1 = P$$

$$21\tfrac{1}{2}V = P$$

$$14\tfrac{1}{3}V^1 = 21\tfrac{1}{2}V$$

$$\tfrac{43}{3}V^1 = \tfrac{43}{2}V$$

$$86V^1 = 129V$$

$$V^1 = \tfrac{129}{86}V$$

$$V^1 = 1.5V$$

$$25\tfrac{1}{3}(1.5V) = x + 1 \quad \text{Substituting } 1.5V \text{ for } V^1$$

$$38V = x + 1$$

$$38V = 36V + 1 \quad \text{Substituting } 36V \text{ for } x$$
$$\therefore 2V = 1$$
$$V = .5$$
$$V^1 = 1.5 \times .5 = .75$$
$$36 \times .5 = 18$$
$$25\tfrac{1}{3} \times .75 = 19$$
$$A = 19'' \text{ long}$$
$$B = 18'' \text{ long}$$

The above solution is similar to Solution V. The girl who presented this was able to see that this was a problem involving rate and time and proceeded to find the rate of burning for each candle. It was then a simple task to find the length of the candles. This pupil is taking courses in science as well as in mathematics and like the boy who presented No. V was influenced in her solution of the problem by her training in science.

Solution VIII

Let x = time A burns minus the time it takes A to burn one inch.
860 = number of minutes between time they burn even and time B burns out.
1290 = number of minutes between time they burn even and time A burns out.
36 = number of hours B burns.

$$\therefore \frac{860}{1290} = \frac{x}{36}$$
$$x = 24 \text{ hours}$$

$25\tfrac{1}{3}$ hrs. = no. of hrs. A burns
$25\tfrac{1}{3} - 24 = 1\tfrac{1}{3}$ hrs.—no. of hrs. it takes A to burn 1 inch
$\therefore 25\tfrac{1}{3} \div 1\tfrac{1}{3} = 19''$ length of A
$19 - 1 = 18''$ length of B

In this solution the time relationship is taken into consideration and the student has used the principle of proportion to aid him in solving the problem.

Summary

1. An examination of the variety in these solutions presented above shows clearly that pupils trained under a plan in which the mathematical subjects are correlated possess power of resourcefulness.

2. It further shows that the pupils have assimilated the subject matter and methods to the extent that they turn out to be a valuable tool in a new situaion.

3. I also showed desirability of correlating mathematical subjects in the Junior High School as it makes a better appeal to the pupil's interest in arithmetic which is tiresome to the young boy or girl and frequently destroys all incentive to continue the study of mathematics.

4. The problem also aroused a great deal of interest in mathematics, as many pupils not enrolled in courses in mathematics spent a number of hours trying to find a solution, and some of these handed in correct solutions.

5. There has been a great deal of discussion pro and con among teachers of mathematics concerning the puzzle problem. Some leading psychologists maintain that problems should be found which not only stimulate the pupil to reason, but also direct his reasoning in useful channels and reward it by results that are of real significance. They argue that reasoning sought for reasoning's sake alone is too wasteful an expenditure of time and is very likely to be inferior as reasoning. In the same breath they admit that these puzzle problems have value as drills in analysis of a situation into its elements that will amuse the gifted children, and as tests of certain abilities. On the whole the contention is that the ordinary problems which ordinary life proffers seem to be the sort that should be reasoned out.

On the other hand, it is stated by some authorities that arithmetic satisfies the puzzle instinct or instinct of curiosity. It is true indeed that children find delight in pitting their skill against a puzzle problem, provided they feel that the solution is within their capabilities. The subjects of factoring, drills in multiples, finding the least common denominator, or the greatest common divisor can be made interesting to youngsters by presenting them in the light of puzzles. If this instinct is once aroused, pupils give dynamic interest and sustained attention even to purely mechanical processes in arithmetic. Many people who have reached years of discretion and whose thoughts usually go to the serious affairs of life invest their energies in attempting to find the solutions of many puzzle problems. Whenever a puzzle problem is inserted in the daily paper of a large city, it is a known fact that thousands of people spend many hours trying to find a solution.

The writer does not advocate the constant use of the puzzle problem in the teaching of arithmetic or in the teaching of

mathematics, but he is convinced that we are going to the extreme in excluding entirely this type of problem. A simple means of attaining motivation is to appeal to any one of the pupil's impelling instincts. Thus many of the mechanical processes of mathematics are made natural by appealing to the puzzle instinct, the urge to overcome obstacles the conquest of which seems to be within their capacities. Pupils find that these processes are not stupid when they are presented in the light of puzzles that challenge their ingenuity. Thus to neglect this type of problem is to reduce the study of mathematics to a mechanical and arbitrary series of symbols and processes devoid of life and interest.

NEWS AND NOTES

The annual meeing of the Inland Empire Council of Teachers of Mathematics, which was held at Spokane April 4th and 5th, was voted by all who attended it the most successful and interesting session ever held. Since its reorganization three years ago under the present name and plan of organization it has been rapidly expanding in interest and numbers over the field it attempts to cover, the four northwest states of Oregon, Washington, Idaho, and Montana.

The special features of the meetings were the two excellent and thought-provoking addresses of Professors Griffin and Bratton. The complete program was as follows:

1. "The Role of Mathematics in Human Progress," Prof. Frank L. Griffin, Reed College, Portland, Ore.

2. Report of Committee on "A Course of Mathematical Reading for Teachers." Chairman, Prof. C. A. Isaacs, Washington State College, Pullman, Wash.

3. "Report of Committee on Applications of the Report of the National Committee on Mathematical Requirements to Conditions in the Northwest." Chairman, Winona Perry, Coeur d'Alene High School, Coeur d'Alene, Idaho.

4. "What a High School Teacher of Mathematics Ought to know About Einstein," Prof. Walter A. Bratton, Whitman College, Walla Walla, Wash.

5. Round Table Discussion—"Practical Methods in Teaching." (a) In Arithmetic—Home Work; Local Problems; Intuitional Geometry; Literal Arithmetic. (b) In Algebra—Factoring; Negative Numbers; Thought Problems; Graphs; Determinants. (c) In Geometry—Originals; Theory of Limits; Use of Colored Crayons; Use of Models; Evaluation of Pi.

In place of Professor Walter C. Eells of Whitman College, who has been President of the Council for three years, W. W. Jones, head of the mathematics department in the North Central High School of Spokane, was elected President. Miss Olive Fisher, as Secretary, was succeeded by Miss Christine Claussen of Lewis & Clarke High School, Spokane. Prof. C. A. Isaacs of W. S. C., Pullman, Wash.; Ida A. Mosher, North Central High School, Spokane, and J. C. Teeters, Superintendent, Kellogg, Idaho were elected members of the Executive Committee.

The Summer Session of the Massachusetts Institute of Technology offers two special courses for teachers in mathematics: (1) History of Mathematical Science (special course for teachers), and (2) Differential and Integral Calculus (special course for teachers). Both courses are to be given by Harry W. Tyler, Ph.D., Walker Professor of Mathematics.

Other attractive courses are offered in Chemistry and Physics.

The Association of Teachers of Mathematics in New England held an informal dinner program April 7, 1923, in Cambridge. The subject of the discussion was, "The Advantages of a Course in General Mathematics for the First Two Years in High School and for the First Year in College."

The speakers were: Professor F. S. Woods, Massachusetts Institute of Technology; Professor George D. Birkhoff, Harvard University; Professor William R. Ransom, Tufts College, and Mr. Louis A. McCoy, English High School, Boston.

The ninth meeting of the Association of Teachers of Mathemtics of North Carolina was held at the University at Chapel Hill, N. C., on February 16-17, 1923.

On Friday evening, after words of welcome by Dr. Henderson on behalf of the university, Mr. W. S. Schlauch of the High School of Commerce, New York City, addressed the association on "Practical Mathematics in the High School." In this message Dr. Schlauch emphasized the fact that if pupils are to be interested in mathematics, they must be shown that it really functions in life, and also if they are to attack problems with a zest, they must feel that they are accomplishing a result or solving a real situation.

A resolution that the association consider themselves members of the National Association was adopted, and the members were urged to subscribe to THE MATHEMATICS TEACHER.

The motion to the effect that a committee be appointed to map out a course of study of mathematics for the high schools of North Carolina was carried. It was suggested that the members of this committee each formulate a course of study, that they then meet in some central point, discuss the various courses and formulate one which they should within the year submit to the association for approval.

The secretary was authorized to send resolutions to the Legislature endorsing the Higher Education Bill, going on record as favoring the full appropriation for educational purposes.

The secretary was also asked to write to Dr. David Eugene Smith of Columbia University a note of appreciation for the good wishes sent by him to the association.

The following persons were elected: President, S. B. Smithey, Greensboro, N. C.; Vice President, Miss Julia Groves, Salisbury, N. C.; Secretary-Treasurer, Miss Nita Gressit, Greensboro, N. C.

Following the business session, Mr. Schlauch again had the rapt attention of his hearers while he discussed the "Teaching of Geometry to Develop Analytic Methods of Thought." According to Dr. Schlauch pupils should have preliminary training in geometry in grades 7, 8 and 9, and the subject should be organized scientifically in the senior years, where it should be handled analytically, the teaching being a matter of induction.

In the Round Table Discussions, the following subjects were discussed:

"Typical Errors in Elementary Mathematics," led by Mr. J. W. Lasley, Jr., of the University of North Carolina; "Grading," led by Miss Winsted. "Tests and Measurements," led by Mr. Schlauch of New York City.

Miss Mendenhall of the North Carolina College for Women, Greensboro, N. C., agreed to send, upon request, a bulletin of common errors of her own compilation, to any one present.

JUANITA PUETT, *Secretary*,
Laurinburg, N. C.

The Association of Mathematics Teachers of New Jersey held the eighteenth regular meeting in Newark on May 12, 1923.

The program consisted of: (1) "The Elements of a Composite Examination in Plane Geometry," by Miss Vera Sanford, of the Lincoln School; (2) "A First Course in Analysis," by Professor W. R. Longley, of Yale University; (3) Report of Committee on Examinations, by Howard F. Hart, Chairman, of the Montclair High School. Commissioner Lambert L. Jackson led the discussion.

The officers of the society are: P. W. Averill, President, Battin High School, Elizabeth, N. J.; H. F. Hart, Vice President,

Montclair High School; Andrew S. Hegeman, Secretary-Treasurer, Central High School, Newark, N. J.

According to the judgment of the executive department of the Detroit School system "The Detroit Mathematics Club" is the most successful organization among the clubs in the Detroit Public Schools. The club held its fourth and final meeting for 1922-1923 on April 19, 1923.

The following officers were elected for the year 1923-1924: President, Miss Helen Irland, Eastern High School; Vice-President, Mr. Orrin Seaver, Southwestern High School; Secretary, Mrs. Mabel Boville, Northwestern High School; Treasurer, Mr. Wm. Thompson, Hutchins Intermediate School.

Count Korzybski's address on Jan. 11, 1923 will soon appear in "The Mathematics Teacher."

Prof. W. B. Ford, of the University of Michigan on March 8, 1923, gave us an illustrated lecture on "Mathematical Historical Pisa." This talk and slides gave us the proper perspective from which to view the famous tower, as well as making us see it as only a part of the University of Pisa, one of the oldest universities on the continent of Europe.

At our first meeting on April 19, 1923, Mr. E. L. Miller (Supt. of the Detroit High Schools) gave us a very interesting and amusing talk on "Schoolmasters," from the time before "King Tut" to the present day, as they are pictured in fiction, past and present. He claimed that all really great teachers are written into *some* book, by *some one*, at *some* time. The Schoolmaster of the past was pictured as ruling with the rod as a means of preventing the spoiling of the child. In this way that magic thing called "discipline" was maintained.

In closing, the speaker said that our problem as teachers really was "The Human Equation." (Selah Warren Mullen, Northwestern High School)

NEW BOOKS

High School Mathematics: A First Course. By John A. Swenson. The Macmillan Co., New York, 1923.

This very readable book is intended primarily as a text in general mathematics for the ninth year, although sufficient subject matter is presented for more advanced work. It comprises work in algebra, intuitive and demonstrative geometry, trigonometric ratios and a peep into analytical geometry in the chapter on graphic solutions of equations.

The work is so arranged as to correlate algebraic and geometrical interpretations and yet to permit a different sequence of topics if such be preferred. Very little attempt is made to motivate the work but the development of much of the new is pleasing and two pages of brief historical notes on algebra and geometry add a touch of interest.

The selection of material for such a text and its method of presentation must be determined by the three general purposes of mathematical instruction for ninth year pupils—utility, discipline and appreciation.

Let us first consider *utility.* The pupil should be trained in the art of computation; he should be acquainted with commercial forms and business practices; he should have a good idea of geometrical measurements and should understand such relationships as are presented in the use and interpretation of formulas, equations and graphs.

In this text, practice in computation is afforded in the work on statistical graphs, in computing arithmetic averages and means, in involution and evolution, in operations with numbers of direction, in the use of logarithms, in approximate computations, in the trigonometric ratios, in variation, and in ratio and proportion. Frequent use is made of decimal and per cent expressions throughout the exercises, and the use of tables is a prominent feature.

Commercial forms are not touched upon. The application of the formula to business arithmetic is treated in the supplementary chapter.

Two full chapters are devoted to intuitional geometry.

Great stress is laid upon the use of formulas. The treatment of the equation is interesting and valuable. Much is done in the way of interpretation and application. The quadratic is reduced to the form $x^2 + bx = c$ and a mechanical rule is developed for solving without completing the square. Simultaneous quadratics are discussed in the supplementary chapter.

With regard to the *disciplinary value,* as much material as could be desired for ninth grade work in demonstrative geometry is given. Three chapters are devoted to this phase of the work. They contain the proofs of thirty-five propositions and over two hundred exercises. Rigor in establishing algebraic principles is lacking.

No attempt is made to teach the technique of formulating definitions or of proving principles.

The problem material is interesting, varied, thought-provoking and educational, affording abundant opportunity for the cultivation of accuracy of thought and for the exercise of judgment. Constant appeal is made to the intelligent interpretation of symbols and of relations.

The *appreciative* appeal is not lacking. It is difficult to see how a class using these exercises and developments under the guidance of a skilful teacher could fail to feel enthusiasm and admiration for this classic subject and to appreciate, in some slight degree at least, its compelling fascination.

Mary S. Taylor,
New York Training School for Teachers.

Practical Trade Mathematics for Electricians, Machinists, Carpenters, Plumbers, and Others. By James A. Moyer and Charles H. Sampson, B. S., New York. John Wiley & Sons, Inc., 1920, pp. 172.

This book is intended for adult artisans. The arrangement of topics is conventional, beginning with the four fundamental operations with integers, followed by common fractions, decimal fractions, percentage, ratio and proportion, and so on. Most of the book is devoted to arithmetic, including mensuration of surfaces and solids. There are some formulas and a very brief treatment of graphs.

Problems at the end of each chapter are arranged in two groups. "The first group is intended to include general ap-

plications of mathematics, especially of machinists, draftsmen, plumbers, loom fixers, carpenters, steam engineers, firemen and sheet metal workers. The second group is especially for students engaged in electrical trades."

The impossibility of writing practical problems for the wide variety of trades mentioned under group one is at once evident. What is practical for the carpenter may have only academic interest for the loom fixer.

This book is one of a fairly large number of text-books in mathematics which have appeared within the last few years claiming on title page or in preface to be "Practical." In the present book practical, if one may judge by many of the problems offered, is an elastic term. One would suppose it to mean problems such as might arise in the practice of a given trade. It is doubtful whether enough such problems can be found to fill a book. It is typical of many of these books that the problems offered are not by any means such as would ever arise, but have been prepared so as to give a fictitious appearance of reality. One would welcome a more accurate statement of the situation. The following two are typical, of not all but many, of the problems.

> A certain wall exerts a pressure on the soil of 225 pounds per square foot. What pressure would this be in ounces per square inch?
>
> A piece of bare copper wire weighs $\frac{9}{16}$ ounce per inch. How many inches will weigh 15 21-256 ounces?

Men engaged in the trades mentioned in the preface are constantly using numbers obtained from measurement. In computing, these men need guidance in regard to the reliability of such numbers. This treatment can be simple, but accurate enough for their needs. Such subjects as significant figures, contracted forms of multiplication, and division, the use of the slide rule, and possibly the use of logarithms should appear. They are not found in this book. Failure to keep the idea of significant figures in mind leads to such a problem as: "Find the square root of 89526.025681," eleven significant figures.

The authors call attention to the importance of checking, but give some unfortunate illustrations. Divisions, for instance, is not a common form of check for multiplication. The check given for division, multiplying divisor by quotient, is quite wrong. There appears no mention of the valuable device the preliminary estimate, rough check.

Many of the illustrations are trivial, being of no help in the solution of the problem. A curious instance is a problem dealing with the increase in price of wire nails, for which is supplied a picture of a wire nail.

There are some slips which mar the book: the definition of division on page 4; the statement in regard to ratio of areas on page 77; the note of advice with problem 31, on page 164.

John W. Regan,
Charlestown High School, Boston.

The Teaching of Arithmetic. By N. J. Lennes. The Macmilan Company, New York, 1923.

In a book of nearly five hundred pages Professor Lennes devotes one hundred seventy pages to general method and educational theory. In his first paragraph he says: "This will serve to clear the ground and lay a basis for the discussions which comprise the greater part of this book." So we have the following chapters, "Formal Discipline," "Methods of Learning and Teaching," "Motivation in the Early Grades," and "Motivation in the Grammar Grades."

Relative to formal discipline the author takes the view that while the workers in psychological laboratories have contributed much to the cause of education, there are qualities and manifestations of mind which they have been unable to deal with, and these are the very qualities which are of highest value in human life. Methods of learning outside of school are quite different from those used in school, and the attempt of school people to make them the same is unwise. To quote, "One cannot escape the feeling that . . . there has been a yielding to the pressure to make life in the school similar to the majority of those outside of it, a life of much action and little thought. If school is to prepare truly for life, it must somehow prepare for a life of which thinking shall be a relatively large ingredient. But how?" And the answer, implied throughout the book, is that we must have teachers who *can* think and *like* to think.

The chapters on motivation are critical of some things that pass for motivation; without troubling to formulate definitions, he takes the view that the value of the project method is that it makes for proper motivation. Many excellent illustrations of projects are given.

Part II, "Special Problems in the Teaching of Arithmetic," includes a chapter on "Objects to be Attained in Arithmetic," then twenty chapters (242 pages) devoted to as many arithmetical topics ranging from "Numbers and Number Combinations" to "Mensuration." The treatment of these topics is mathematically sound, and on the pedagogical side avoids the errors of those who strain at a "foolish consistency." Then follow chapters headed "The Arithmetic of the Home," "Arithmetic for the Farm," "The Textbook, Its Selection and Use," and (inevitably) "Measuring the Results of Teaching."

I cannot recall ever having read a book on the teaching of arithmetic that gave me more pleasure than this one. The style is distinctly literary and delightfully discursive; one wonders if the author, even once, resisted his desire to turn aside at a corner of suggestion and traverse the by-way of thought to its end—and back again.

But a reviewer *must* criticize something *adversely*. The book seems to be addressed to the teacher of some experience, for it falls short of giving the specific and detailed help so much needed by the normal school student and the "green" teacher. It would seem that the things of greatest *immediate* importance to the beginner are often so completely surrounded by an alluring environment of humor and homely philosophy as to lose their proper emphasis. The teacher of method might well urge his students to read Professor Lennes' book at their first opportunity, but, considering the time usually allowed for courses in methods in arithmetic, he might hesitate to use the book as a class text.

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